Computability Final Test — 2014-09-02

Notes.

- Answer both theory questions, and choose and solve two exercises, only. Solving more exercises results in the failure of the test.
- To pass the exam you need to provide a reasonable contribution in *both* Theory and Exercises parts.
- Exercises with higher number award more points. To achieve a score ≥ 28 you have to solve an exercise marked with $\star$ below.
- Significantly wrong answers will result in negative scores.
- Always provide a justification for your answers.

Reminder: when equating the results of partial functions (as in $\phi_i(3) = \phi_i(5)$), we mean that either 1) both sides of the equation are defined, and evaluate to the same natural number, or 2) both sides are undefined.

Theory

Question 1. *Prove that the Kleene set* K *is not recursive.*

Question 2. *Define the set of recursive partial functions* $\mathcal{R}$*. This definition involves several operators to construct a partial function from other ones. Provide a definition for these operators, too.*

Exercises

Exercise 3. *Prove whether*

$$A = \{n \mid (\forall x \in \mathbb{N}.\ \exists y \in \mathbb{N}.\ \phi_n(2 \cdot x) = 2 \cdot y) \land \phi_n(5) = undefined\} \in \mathcal{RE}$$

Solution (sketch). The set A is semantically closed (easy to check). Let $\mathcal{F}$ be its associated set of functions. We prove $A \notin \mathcal{RE}$ by Rice-Shapiro ($\Rightarrow$).

Take $f(x) = x$ for x even, and undefined for x odd. Function f is clearly recursive (the test for evenness is recursive, and the two branches are recursive). We have $f \in \mathcal{F}$, since for each x there exists $y = x$ for which $f(2 \cdot x) = 2 \cdot y$. Moreover $f(5)$ is undefined.

Now, let g be any finite restriction of f. So $g(x) = undefined$ as long as x is large enough, say $x \geq k$ for some threshold k. Take $x = k$ if k is even, otherwise $x = k+1$. In any case, x is even, and we have $g(x) = undefined \neq 2 \cdot y$ for any y. Hence, $g \notin \mathcal{F}$. $\square$

Exercise 4. *Given an arbitrary partial function f, define*

$$A_f = \{n \mid f(n) = 5\}$$

1. *Show that, for any total recursive f, the set A_f is $\mathcal{R}$.*

2. *Then, construct a partial recursive g such that A_g is $\mathcal{RE}$ but not $\mathcal{R}$.*

3. *Finally, construct a total function h such that A_h is not $\mathcal{RE}$.*

Solution (sketch). Given any total recursive f, we can write a verifier for A_f in pseudo-code as follows:

$$
\begin{array}{l}
V_{A_f}(n): \\
\quad x := f(n) \\
\quad \mathsf{if}\ x = 5\ \mathsf{then} \\
\quad\quad \textit{return}\ 1; \\
\quad \mathsf{else} \\
\quad\quad \textit{return}\ 0;
\end{array}
$$

The line $x := f(n)$ is well-defined since f is recursive, so there is some way to implement it. Since f is total, that line will always halt. So, whenever $n \in A_f$ variable x will get assigned 5 and the whole program V_{A_f} will return 1. Otherwise, when $n \notin A_f$ variable x will get assigned some natural $\neq 5$, hence V_{A_f} will return 0.

Take $g(n) = 5 \cdot \tilde{\chi}_{\mathsf{K}}(n)$, which is recursive since it is a composition of recursive functions (and we know that $\tilde{\chi}_{\mathsf{K}}$ is recursive). We have then $A_g = \mathsf{K}$ which is $\mathcal{RE}$ but not $\mathcal{R}$.

Take $h(n) = 5 \cdot \chi_{\bar{\mathsf{K}}}(n)$. This is total because $\chi_{\bar{\mathsf{K}}}(n)$ is always defined. The set A_h is $\bar{\mathsf{K}}$ which is not $\mathcal{RE}$. □

Exercise 5. *Prove whether*

$$A = \{n \mid \mathsf{ran}(\phi_n) \textit{ is infinite}\} \leq_m \{n \mid \mathsf{dom}(\phi_n) \textit{ is infinite}\} = B$$

Solution (sketch). Take

$$h(n) = \# \left(\lambda x. \begin{cases} 0 & \text{if } \exists y.\ \phi_n(y) = x \\ \textit{undefined} & \text{otherwise} \end{cases} \right)$$

The guard is $\mathcal{RE}$ (easy to check: existential quantifier applied to the $\mathcal{RE}$ predicate $\phi_n(y) = x$.). Since 0 is recursive and the other branch is undefined, by the if-then-else lemma h is well defined and total recursive.

Note that $\exists y.\ \phi_n(y) = x$ is equivalent to $x \in \mathsf{ran}(\phi_n)$. So, $\phi_{h(n)}(x)$ is defined (as 0) exactly when x belongs to $\mathsf{ran}(\phi_n)$. Hence $\mathsf{dom}(\phi_{h(n)}) = \mathsf{ran}(\phi_n)$, from which it immediately follows that h m-reduces A to B.

□

Exercise 6. $\star$ *Let A, B be two arbitrary sets $A \subseteq B \subseteq \mathbb{N}$ such that $B \setminus A$ is infinite. Prove that there is a set $C \notin \mathcal{RE}$ such that $A \subseteq C \subseteq B$.*

Solution (sketch). Intentionally omitted. □