Computability Final Test — 2014-02-03

Notes.

- Answer both theory questions, and choose and solve two exercises, only. Solving more exercises results in the failure of the test.
- To pass the exam you need to provide a reasonable contribution in *both* Theory and Exercises parts.
- Exercises with higher number award more points. To achieve a score ≥ 28 you have to solve an exercise marked with $\star$ below.
- Significantly wrong answers will result in negative scores.
- Always provide a justification for your answers.

Reminder: when equating the results of partial functions (as in $\phi_i(3) = \phi_i(5)$), we mean that either 1) both sides of the equation are defined, and evaluate to the same natural number, or 2) both sides are undefined.

Theory

Question 1. *Prove that, if $A, B \in \mathcal{RE}$, then $A \cup B \in \mathcal{RE}$ and $A \cap B \in \mathcal{RE}$. Also prove that, under the same assumption, $A \setminus B \in \mathcal{RE}$ does not hold in general.*

Question 2. *State the Rice-Shapiro theorem. Then, prove its ($\Rightarrow$) direction.*

Exercises

Exercise 3. *[70% score] Prove whether $A \in \mathcal{RE}$, where:*

$$A = \{n \mid \phi_n(100) = 10 \wedge \forall x \in \mathbb{N}.\ \phi_n(2 \cdot x) = 2 \cdot \phi_n(x)\}$$

[30% score] Then, prove whether $B \in \mathcal{RE}$, where:

$$B = \{n \mid \forall x \in \mathbb{N}.\ \phi_n(2 \cdot x) = 2 \cdot \phi_n(x)\}$$

Solution (sketch). $A = \emptyset \in \mathcal{RE}$. Indeed, if $n \in A$, we have $10 = \phi_n(100) = 2 \cdot \phi_n(50) = 2 \cdot 2 \cdot \phi_n(25)$ which is impossible: if $\phi_n(25)$ is undefined then 10 should also be undefined, otherwise if $\phi_n(25)$ is defined then 10 should be a multiple of 4.

For the second part, $B \notin \mathcal{RE}$ by Rice-Shapiro ($\Leftarrow$). Taking $g(n) = undefined$ and $f(n) = 1$ suffices since $1 = f(2 \cdot 1) \neq 2 \cdot f(1) = 2$. $\square$

Exercise 4. *[80% score] Prove whether $\mathsf{K} \leq_m C = \{n \mid \mathsf{dom}(\phi_n) \in \mathcal{R}\}$.*
[20% score] Then, also prove whether $\bar{\mathsf{K}} \leq_m C$.

Solution (sketch). $\mathsf{K} \leq_m C$ because the following is a reduction

$$h(n) = \#\left(\lambda x. \begin{cases} 0 & \text{if } n \in \mathsf{K} \vee x \in \mathsf{K} \\ undefined & \text{otherwise} \end{cases}\right)$$

The body of the λx is recursive by the if-then-else lemma with $\mathcal{RE}$guard (because...). So by the s-m-n theorem, h is well-defined and recursive total. Then:

- If $n \in \mathsf{K}$, then $\phi_{h(n)}(x) = 0$ for all x, then $\mathsf{dom}(\phi_{h(n)}) = \mathbb{N} \in \mathcal{R}$, so $h(n) \in C$.
- If $n \notin \mathsf{K}$, then $\phi_{h(n)}(x) = 0$ for $x \in \mathsf{K}$ and undefined otherwise, then $\mathsf{dom}(\phi_{h(n)}) = \mathsf{K} \notin \mathcal{R}$, so $h(n) \notin C$.

$\bar{\mathsf{K}} \leq_m C$ because the following is a reduction

$$h(n) = \#\left(\lambda x. \begin{cases} 0 & \text{if } n \in \mathsf{K} \wedge x \in \mathsf{K} \\ undefined & \text{otherwise} \end{cases}\right)$$

As before, h is well-defined and recursive total. Then:

- If $n \in \bar{\mathsf{K}}$, then $\phi_{h(n)}(x) = undefined$ for all x, then $\mathsf{dom}(\phi_{h(n)}) = \emptyset \in \mathcal{R}$, so $h(n) \in C$.
- If $n \notin \bar{\mathsf{K}}$, then $\phi_{h(n)}(x) = 0$ for $x \in \mathsf{K}$ and undefined otherwise, then $\mathsf{dom}(\phi_{h(n)}) = \mathsf{K} \notin \mathcal{R}$, so $h(n) \notin C$.

□

Exercise 5. *Prove whether $f \in \mathcal{R}$, where:*

$$f(n,x) = \begin{cases} 2 \cdot n & \textit{if } \phi_n(x) \textit{ halts in } \geq x \textit{ steps} \\ 51 & \textit{otherwise} \end{cases}$$

Solution (sketch). We have $f \notin \mathcal{R}$. Assume by contradiction $f \in \mathcal{R}$. Since f is total, the set $A = \{n \mid f(n,0) = 2 \cdot n\}$ is recursive. However, since 51 is odd and $2 \cdot n$ is even for any n, we have $A = \{n \mid \phi_n(0) \text{ halts in } \geq 0 \text{ steps}\} = \{n \mid \phi_n(0) \text{ is defined}\}$. We reach a contradiction since $A \notin \mathcal{R}$ by Rice. □

Exercise 6. ⋆ *Define an infinite sequence of sets $A_0, A_1, A_2, \dots$ such that, for all $i \in \mathbb{N}$:*

$$A_i \subseteq A_{i+1} \qquad \wedge \qquad A_i \in \mathcal{RE} \iff i \textit{ is even}$$

Hint: can $A_{i+1} \setminus A_i$ be finite? Construct the sets accordingly.

Solution (sketch). Intentionally omitted. □