Mathematical Logic - 2017

Propositional Logic: exercises

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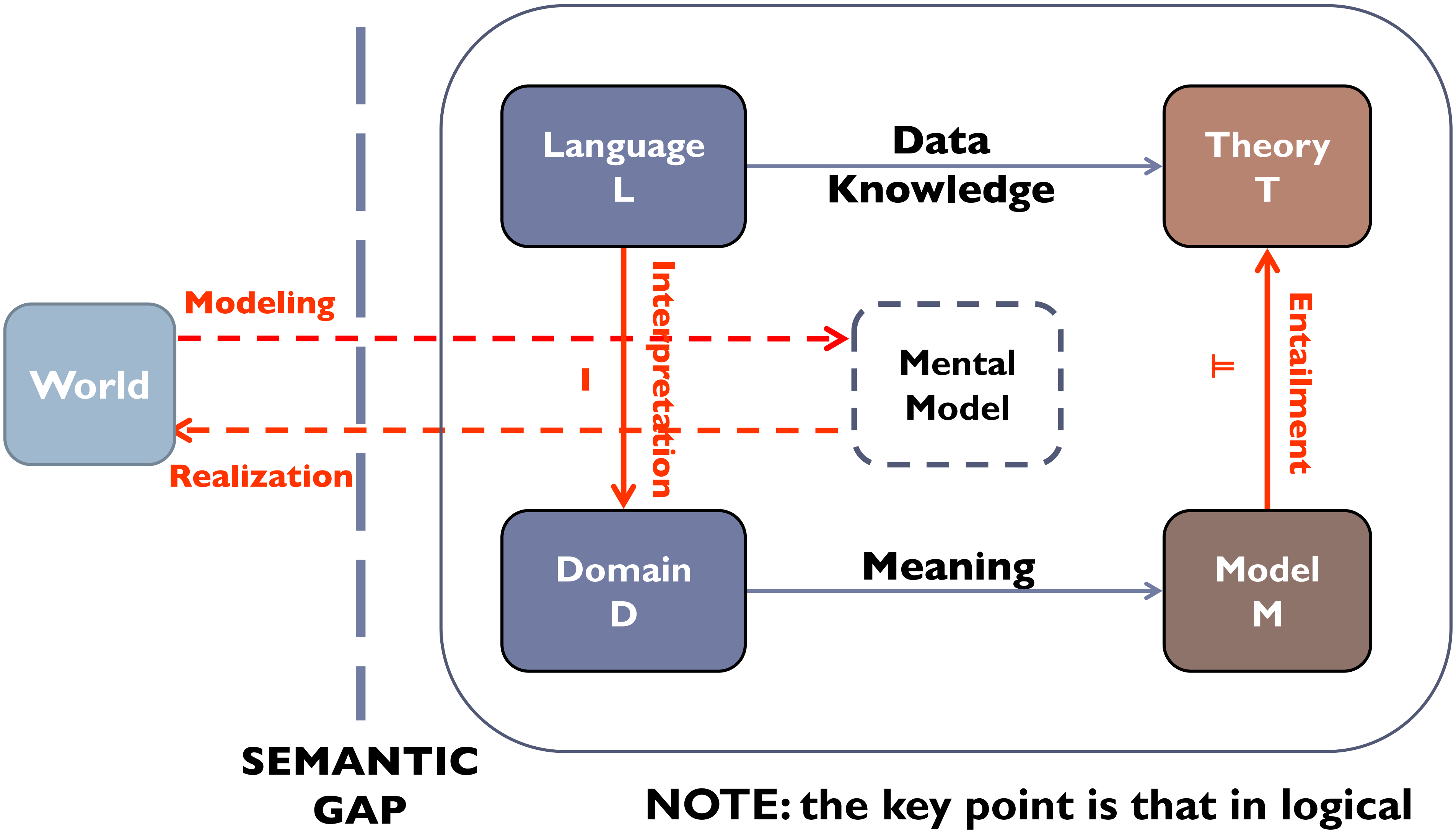
Logical Modeling



NOTE: the key point is that in logical modeling we have formal semantics

Modeling Exercise: Forest

- **Description**: There are two lions, Kimba and Simba, in the forest. They are in competition for the food. There is a nice antelope they want to hunt. If they want to survive they have to catch it.

- **Problem**: Model the problem by identify relevant objects, defining the domain, the language, the theory and providing a possible intentional model.

Solution: Forest (I)

- **Description**: There are two lions, Kimba and Simba, in the forest. They are in competition for the food. There is a nice antelope they want to hunt. If they want to survive they have to catch it.

Relevant objects are in red

D = {T, F}

L = {Lion, Antelope, Survive, Catch}

Solution: Forest (II)

- The theory T:

  $Antelope \wedge (Catch \vee \neg Survive)$

  $Antelope \vee \neg Catch$

- An interpretation:

I (Lion)	= T	I (Antelope)	= T
I (Catch)	= T	I (Survive)	= T

- I is a model for T

- I below is NOT a model for T

I (Lion)	= T	I (Antelope)	= F
I (Catch)	= F	I (Survive)	= T

Modeling Exercise: Classroom

- **Description**: In a class there are several persons. Usually there is one professor who teaches to some students. Students can be Master students or PhD students. Among PhD students there might be some Assistants of the professor.

- **Problem**: Model the problem by identify relevant objects, defining the domain and the language, and providing a possible <u>extensional</u> model for it.

Solution: Classroom (I)

- **Description**: In a class there are several persons. Usually there is one professor who teaches to some students. Students can be Master students or PhD students. Among PhD students there might be some Assistants of the professor.

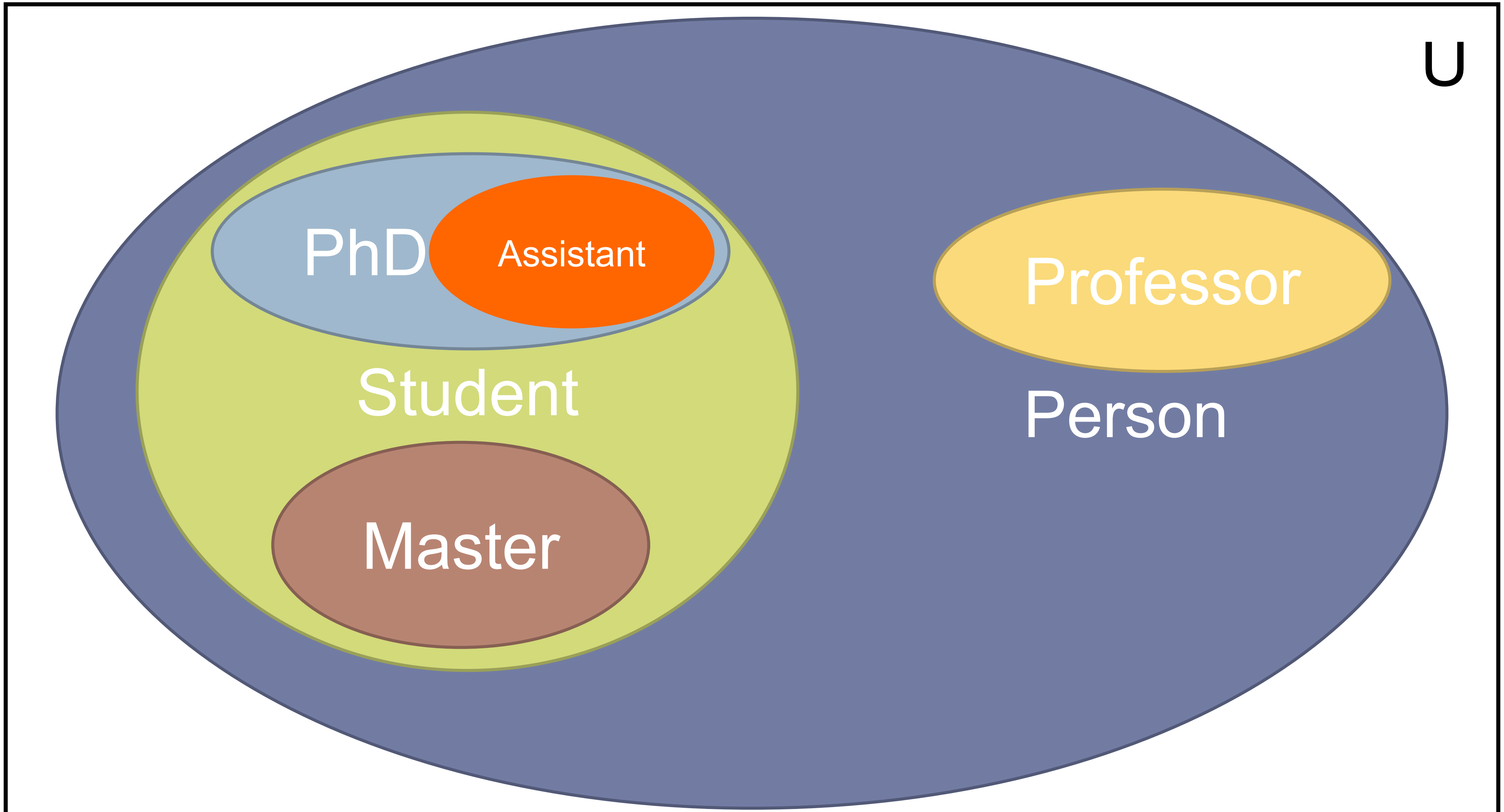
Relevant objects are in red

L = {Person, Professor, Student, Master, PhD, Assistant}

D = {Fausto, Mary, Paul, Jane}

Solution: Classroom (II)

- The corresponding Venn diagram



Solution: Classroom (III)

- A possible model:

I (Person)	= {Fausto, Mary, Paul, Jane}
I (Professor)	= {Fausto}
I (Student)	= {Mary, Paul, Jane}
I (Master)	= {Mary}
I (PhD)	= {Paul, Jane}
I (Assistant)	= {Paul}

Modeling Exercise: Family

- **Description**: My family consists of several members. There is a grandparent and my parents. Then there are some children, i.e. two sisters, one brother and me

- **Problem**: Model the problem by identify relevant objects, defining the domain and the language, and providing a possible <u>extensional</u> model for it.

Solution: Family (I)

- **Description**: My family consists of several members. There is a grandparent and my parents. Then there are some children, i.e. two sisters, one brother and me

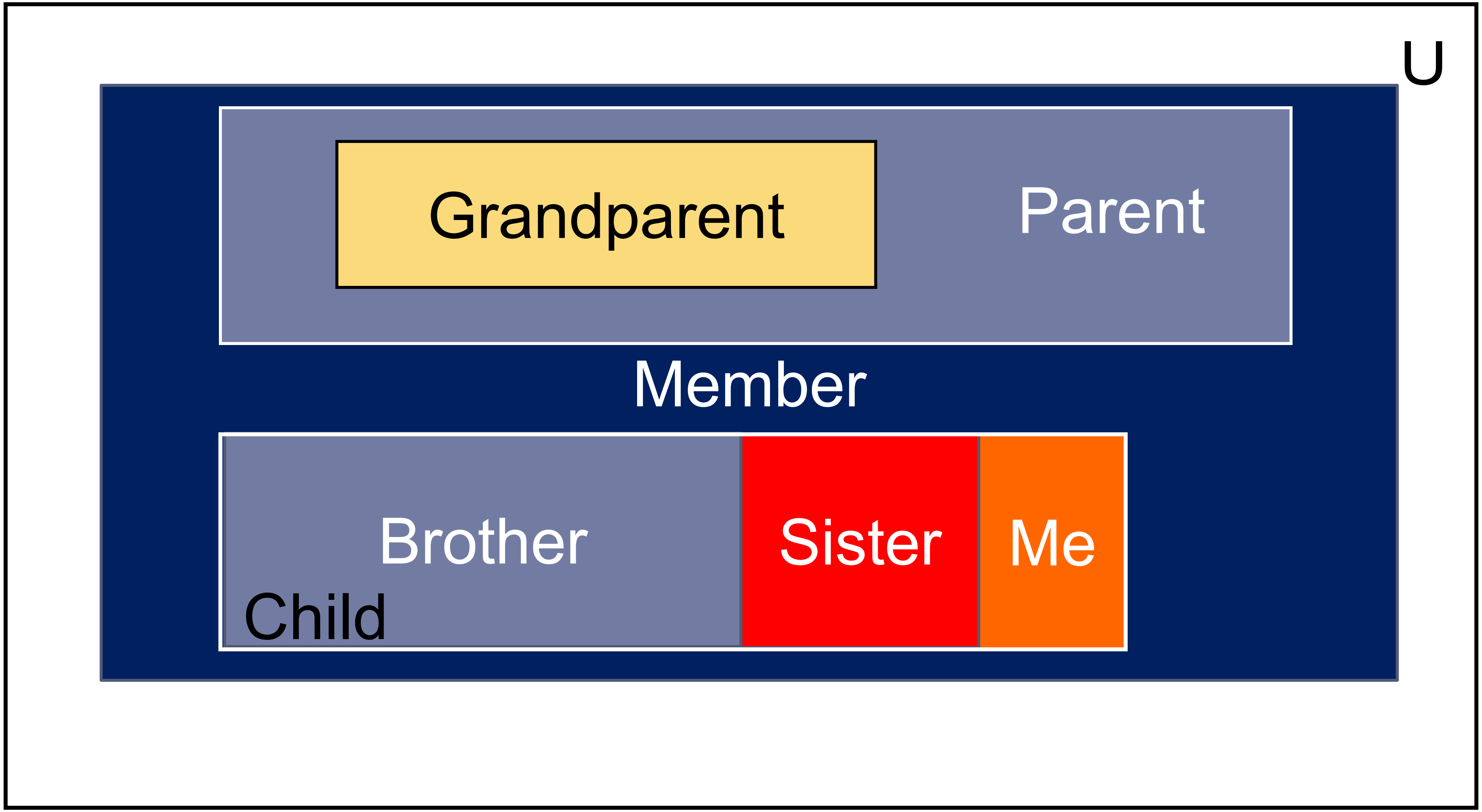
Relevant objects are in red

L = {member, grandparent, parent, child, brother, sister, me}

D = {Bob, Fausto, Mary, Paul, Jane, Hugo, Robert}

Solution: Family (II)

- The corresponding Venn diagram



Solution: Family (III)

- A possible model:

I (Member)	= {Bob, Fausto, Mary, Paul, Jane, Hugo, Robert}
I (Grandparent)	= {Bob}
I (Parent)	= {Fausto, Mary, Bob}
I (Brother)	= {Robert, Paul}
I (Sister)	= {Jane}
I (Me)	= {Hugo}

Modeling Exercise: My friends

- **Description**: I have a lot of friends. I met some of them on the forum of my website. However, only a few of them are close to me. In particular, I use to play chess with Paul.

- **Problem**: Model the problem by identify relevant objects, defining the domain and the language, and providing a possible <u>extensional</u> model for it.

Propositional logic language

Propositional alphabet:

- Logical symbols: $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$
- Non logical symbols A set Ω of symbols called propositional variables
- Separator symbols "(" and ")"
- "Meta-symbols", i.e. $\models$, $\top$ or $\perp$

Definition (Well formed formulas):

- Every P $\in$ Ω is an atomic formula
- Every atomic formula is a formula
- If A and B are formulas then $\neg$A, A $\wedge$ B, A $\vee$ B, A $\rightarrow$ B, e A $\leftrightarrow$ B are formulas

Symbols in PL

Which of the following symbols are used in PL?

$$\sqcap \quad \neg \quad \top \quad \vee \quad \equiv \quad \sqcup \quad \sqsubseteq \quad \longrightarrow \quad \leftrightarrow \quad \bot \quad \wedge \quad \models$$

$$\sqcap \quad \neg \quad \top \quad \vee \quad \equiv \quad \sqcup \quad \sqsubseteq \quad \rightarrow \quad \leftrightarrow \quad \bot \quad \wedge \quad \models$$

Formalizing NL

Let's consider a propositional language where *p* means *"Paola is happy"*, *q* means *"Paola paints a picture"*, and *r* means *"Renzo is happy"*. Formalize the following sentences:

- "if Paola is happy and paints a picture then Renzo isn't happy"

  $(p \wedge q) \rightarrow \neg r$

- "if Paola is happy, then she paints a picture"

  $p \rightarrow q$

- "Paola is happy only if she paints a picture"

  $\neg(p \wedge \neg q)$ which is equivalent to $p \rightarrow q$!!!

Formalizing NL

Exercise

Let $A =$"**Angelo comes to the party**", $B =$"**Bruno comes to the party**", $C =$"**Carlo comes to the party**", and $D =$"**Davide comes to the party**".

Formalize the following sentences:

1. *"If Davide comes to the party then Bruno and Carlo come too"*
2. *"Carlo comes to the party only if Angelo and Bruno do not come"*
3. *"If Davide comes to the party, then, if Carlo doesn't come then Angelo comes"*
4. *"Carlo comes to the party provided that Davide doesn't come, but, if Davide comes, then Bruno doesn't come"*
5. *"A necessary condition for Angelo coming to the party, is that, if Bruno and Carlo aren't coming, Davide comes"*
6. *"Angelo, Bruno and Carlo come to the party if and only if Davide doesn't come, but, if neither Angelo nor Bruno come, then Davide comes only if Carlo comes"*

Formalizing NL

"If Davide comes to the party then Bruno and Carlo come too"

$$D \rightarrow (B \wedge C)$$

"Carlo comes to the party only if Angelo and Bruno do not come"

$$C \rightarrow (\neg A \wedge \neg B)$$

"If Davide comes to the party, then, if Carlo doesn't come then Angelo comes"

$$D \rightarrow (\neg C \rightarrow A)$$

"Carlo comes to the party provided that Davide doesn't come, but, if Davide comes, then Bruno doesn't come"

$$(C \rightarrow \neg D) \wedge (D \rightarrow \neg B)$$

"A necessary condition for Angelo coming to the party, is that, if Bruno and Carlo aren't coming, Davide comes"

$$A \rightarrow (\neg B \wedge \neg C \rightarrow D)$$

"Angelo, Bruno and Carlo come to the party if and only if Davide doesn't come, but, if neither Angelo nor Bruno come, then Davide comes only if Carlo comes"

$$(A \wedge B \wedge C \leftrightarrow \neg D) \wedge (\neg A \wedge \neg B \rightarrow (D \rightarrow C))$$

Truth valuation

- A truth valuation on a propositional language L is a mapping ν assigning to each formula A of L a truth value ν(A), given the domain D = {T, F}

- ν(A) = T or F according to the modeler, with A atomic
- $\nu(\neg A) \quad = T$ iff $\nu(A) = F$
- $\nu(A \wedge B) \quad = T$ iff $\nu(A) = T$ and $\nu(B) = T$
- $\nu(A \vee B) \quad = T$ iff $\nu(A) = T$ or $\nu(B) = T$

- $\nu(\bot) \quad = F$ (since $\bot =_{df} P \wedge \neg P$)
- $\nu(\top) = T$ (since $\top =_{df} \neg \bot$)

Truth valuation

- Two formulas F and G are logically equivalent (denoted with $F \leftrightarrow G$) if for each interpretation $\mathcal{I}$, $\mathcal{I}(F) = \mathcal{I}(G)$.

- Let F and G be formulas. G is a **logical consequence** of F

(denoted with $F \models G$) if each interpretation satisfying F satisfies also G.

- Let F be a formula:
  - F is valid if every interpretation satisfies F
  - F is satisfiable if F is satisfied by some interpretation
  - F is unsatisfiable if there isn't any interpretation satisfying F

Truth valuation and Truth Tables

- A truth valuation on a PL language L is a mapping ν that assigns to each formula P of L a truth value ν(P).
- A truth table is composed of one column for each input variable and one (or more) final column for all of the possible results of the logical operation that the table is meant to represent. Each row of the truth table therefore contains one possible assignment of the input variables, and the result of the operation for those values.

VARIABLES — **LOGICAL OPERATION**

POSSIBLE ASSIGNEMENTS

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

Example

- Calculate the Truth Table of the following formulas:
  (1) $A \wedge B$;
  (2) $A \vee B$;
  (3) $A \leftrightarrow B$.

	VARIABLES		(1)	(2)	(3)
	A	B	$A \wedge B$	$A \vee B$	$A \leftrightarrow B$
POSSIBLE ASSIGNEMENTS	T	T	T	T	T
	T	F	F	T	F
	F	T	F	T	F
	F	F	F	F	T

Provide the models for the propositions

- A truth valuation v is a model for a proposition P iff v(P) = true

- List the models for the following formulas:
  1. $A \wedge \neg B$
  2. $(A \wedge B) \vee (B \wedge C)$
  3. $(A \vee B) \rightarrow C$
  4. $(\neg A \leftrightarrow B) \leftrightarrow C$

A	B	$A \wedge \neg B$
T	T	F
T	F	T
F	T	F
F	F	F

MODEL

Truth Tables Example (1)

Use the truth tables method to determine whether $(p \rightarrow q) \vee (p \rightarrow \neg q)$ is valid.

p	q	$p \rightarrow q$	$\neg q$	$p \rightarrow \neg q$	$(p \rightarrow q) \vee (p \rightarrow \neg q)$
T	T	T	F	F	T
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

The formula is valid since it is satisfied by every interpretation.

Truth Tables Example (2)

Use the truth tables method to determine whether $(\neg p \vee q) \wedge (q \rightarrow \neg r \wedge \neg p) \wedge (p \vee r)$ (denoted with *F*) is valid.

p	q	r	$\neg p \vee q$	$\neg r \wedge \neg p$	$q \rightarrow \neg r \wedge \neg p$	$(p \vee r)$	F
T	T	T	T	F	F	T	F
T	T	F	T	F	F	T	F
T	F	T	F	F	T	T	F
T	F	F	F	F	T	T	F
F	T	T	T	F	F	T	F
F	T	F	T	T	T	F	F
F	F	T	T	F	T	T	T
F	F	F	T	T	T	F	F

There exists an interpretation satisfying *F*, thus *F* is satisfiable.

Truth Tables Example (3)

Use the truth tables method to determine whether $p \wedge \neg q \rightarrow p \wedge q$ is a logical consequence of $\neg p$.

p	q	$\neg p$	$p \wedge \neg q$	$p \wedge q$	$p \wedge \neg q \rightarrow p \wedge q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	**T**	F	F	**T**
F	F	**T**	F	F	**T**

Truth Tables Example (4)

Use the truth tables method to determine whether $p \rightarrow (q \wedge \neg q)$ and $\neg p$ are logically equivalent.

p	q	$q \wedge \neg q$	$p \rightarrow (q \wedge \neg q)$	$\neg p$
T	T	F	**F**	**F**
T	F	F	**F**	**F**
F	T	F	**T**	**T**
F	F	F	**T**	**T**

Truth Tables Exercises

Compute the truth tables for the following propositional formulas:

- $(p \to p) \to p$
- $p \to (p \to p)$
- $p \vee q \to p \wedge q$
- $p \vee (q \wedge r) \to (p \wedge r) \vee q$
- $p \to (q \to p)$
- $(p \wedge \neg q) \vee \neg(p \leftrightarrow q)$

Truth Tables Exercises

Use the truth table method to verify whether the following formulas are valid, satisfiable or unsatisfiable:

- $(p \rightarrow q) \wedge \neg q \rightarrow \neg p$
- $(p \rightarrow q) \rightarrow (p \rightarrow \neg q)$
- $(p \vee q \rightarrow r) \vee p \vee q$
- $(p \vee q) \wedge (p \rightarrow r \wedge q) \wedge (q \rightarrow \neg r \wedge p)$
- $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$
- $(p \vee q) \wedge (\neg q \wedge \neg p)$
- $(\neg p \rightarrow q) \vee ((p \wedge \neg r) \leftrightarrow q)$
- $(p \rightarrow q) \wedge (p \rightarrow \neg q)$
- $(p \rightarrow (q \vee r)) \vee (r \rightarrow \neg p)$

Problem formalization (1)

Formalize the following argument and verify whether it is correct: *"If you play and you study you'll pass the exams, while if you play and don't study you won't pass. Thus, if you play, either you study and you'll pass the exams, or you don't study and you won't pass."*

L = {P, S, E}

P = "you play", S = "you study"; E = "you pass the exam"

1. $(P \wedge S) \rightarrow E$
2. $(P \wedge \neg S) \rightarrow \neg E$
3. $P \rightarrow (S \wedge E) \vee (\neg S \wedge \neg E)$

We need to prove that $1 \wedge 2 \vDash 3$

Use truth tables

Problem formalization (2)

Brown, Jones, and Smith are suspected of a crime.
They testify as follows:

Brown: *“Jones is guilty and Smith is innocent”.*
Jones: *“If Brown is guilty then so is Smith”.*
Smith: *“I'm innocent, but at least one of the others is guilty”.*

Let B, J, and S be the statements “Brown is guilty”, “Jones is guilty”, and “Smith is guilty”, respectively. Do the following:

1. Express the testimony of each suspect as a PL formula.

The three statements can be expressed as $J \wedge \neg S$, $B \rightarrow S$, and $\neg S \wedge (B \vee J)$

Problem formalization (2)

1. Write a truth table for the three testimonies.

	B	J	S	$J \wedge \neg S$	$B \supset S$	$\neg S \wedge (B \vee J)$
(1)	T	T	T	F	T	F
(2)	T	T	F	T	F	T
(3)	T	F	T	F	T	F
(4)	T	F	F	F	F	T
(5)	F	T	T	F	T	F
(6)	F	T	F	T	T	T
(7)	F	F	T	F	T	F
(8)	F	F	F	F	T	F

Problem formalization (2)

Use the truth table to answer the following questions:

(a) Are the three testimonies satisfiable?

Yes, assignment (6) makes them all true

(b) The testimony of one of the suspects follows from that of another. Which from which?

$J \wedge \neg S \models \neg S \wedge (B \vee J)$

(c) Assuming that everybody is innocent, who committed perjury?

Everybody is innocent corresponds to assignment (8), and in this case the statements of Brown and Smith are false.

(d) Assuming that all testimonies are true, who is innocent and who is guilty?

Assuming that all testimonies are true corresponds to assignment (6). In this case Jones is guilty and the others are innocents.

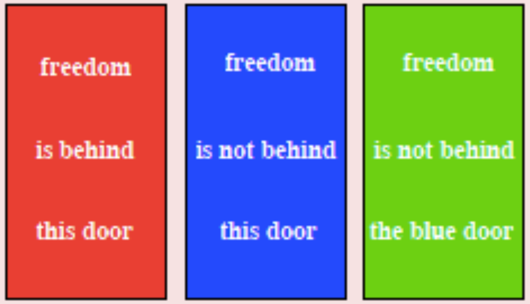
Problem formalization (3)

Problem

Kyle, Neal, and Grant find themselves trapped in a dark and cold dungeon (HOW they arrived there is another story). After a quick search the boys find three doors, the first one red, the second one blue, and the third one green.

Behind one of the doors is a path to freedom. Behind the other two doors, however, is an evil fire-breathing dragon. Opening a door to the dragon means almost certain death.

On each door there is an inscription:



Given the fact that at LEAST ONE of the three statements on the three doors is true and at LEAST ONE of them is false, which door would lead the boys to safety?

Problem formalization (3)

r : "freedom is behind the red door"
b: "freedom is behind the blue door"
g: "freedom is behind the green door"

"behind one of the door is a path to freedom, behind the other two doors is an evil dragon"

$$(r \wedge \neg b \wedge \neg g) \vee (\neg r \wedge b \wedge \neg g) \vee (\neg r \wedge \neg b \wedge g)$$

"at least one of the three statements is true"

$$r \vee \neg b$$

"at least one of the three statements is false"

$$\neg r \vee b$$

Freedom is behind the green door!

r	b	g	2	3	$2 \wedge 3$
T	F	F	T	F	F
F	T	F	F	T	F
F	F	T	T	T	T