Mathematical Logics
Modal Logic: Reasoning*

Fausto Giunchiglia and Mattia Fumagallli

University of Trento

**Originally by Luciano Serafini and Chiara Ghidini*
Modified by Fausto Giunchiglia and Mattia Fumagalli

Proof methods for modal logics

Problem

Problem 1 *How can we show that a modal formula ϕ is valid? (i.e. that $F \models \phi$ for every frame F).*

Problem 2 *How can we show that ϕ is satisfiable? (i.e., that there is a model $M = (F, V)$ and a world $v \in W$ such that $M, w \models \phi$)*

Remark

Problem 1 and problem 2 can be rewriten one in terms of the other. Indeed, proving that $\models \phi$ (i.e., that ϕ is valid) corresponds to prove that $\neg\phi$ is not satisfiable.
Viceversa, proving that ϕ is satisfiable is equivalent to prove that $\neg\phi$ is not valid.

Solution

There are at least two alternatives.

- We can transform ϕ into a first order formula using the standard translation, and to show that ϕ is valid it is enough to show that $\forall x ST^x(\phi)$ is valid.
- we can use a more direct method, and to show that ϕ one can try to search for a counterexample (= an interpretation that falsifies ϕ). and, when trying out all ways of generating a counterexample without success, this counts as a proof of validity. method of (analytic/semantic) tableaux

Reasoning in ML via transformation in FOL

- to check the satisfiability of ϕ_{ML}
- we transform $\phi_{FOL}(x) = ST^x(\phi_{ML})$
- we apply tableaux to $\phi_{FOL}(w)$ for some constant w.

Example

Check if the following formula is valid:

$$(\Box p \wedge \Diamond q) \supset \Diamond(p \wedge q)$$

Solution

- $ST^x((\Box p \wedge \Diamond q) \supset \Diamond(p \wedge q)) =$

$$(\forall y(R(x,y) \supset p(y)) \wedge \exists y(R(x,y) \wedge q(y))) \supset$$
$$\exists y(R(x,y) \wedge P(y) \wedge q(y))$$

- Check if it is valid, e.g., via Tableaux

Reasoning in ML via transformation in FOL

$$\neg(\forall y(R(w,y) \supset p(y)) \wedge \exists y(R(w,y) \wedge q(y))) \supset \exists y(R(w,y) \wedge P(y) \wedge q(y))$$

$$\forall y(R(w,y) \supset p(y)) \wedge \exists y(R(w,y) \wedge q(y))$$
$$\neg \exists y(R(w,y) \wedge P(y) \wedge q(y))$$

$$\forall y(R(w,y) \supset p(y))$$
$$\exists y(R(w,y) \wedge q(y))$$

$$R(w,v) \wedge q(v)$$

$$R(w,v)$$
$$q(v)$$

$$R(w,v) \supset p(v)$$

$$\neg R(w,v) \qquad p(v)$$

CLOSED

$$\neg R(w,v) \wedge p(v) \wedge q(v)$$

$$\neg R(w,v) \qquad \neg p(v) \qquad \neg q(v)$$

CLOSED *CLOSED* *CLOSED*

- The FOL formulas generated by the standard transformation of a modal formulas are of a special forms.
- Quantifiers are always generated in the following two shapes:
  1. $\exists y(R(w,y) \wedge \phi(y))$
  2. $\forall y(R(w,y) \supset \phi(y))$
- γ and δ Tablueaux rules are applied only to these formulas, and generated tableaux of the following two shapes

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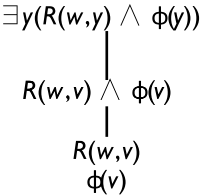

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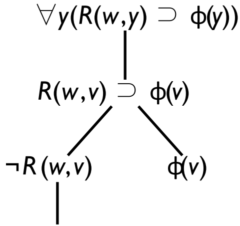


If we have $R(w, v)$ then this branch is closed.
If we don't have $R(w , v)$ this branch will remain open

Analytic/Semantic Tableau Method - References

Early work by Beth and Hintikka (around 1955). Later refined and popularized by Raymond Smullyan:

- R.M. Smullyan. *First-order Logic*. Springer-Verlag, 1968.

Modern expositions include:

- M. Fitting. *First-order Logic and Automated Theorem Proving. 2nd edition.* Springer-Verlag, 1996.
- M. D'Agostino, D. Gabbay, R. Hähnle, and J. Posegga (eds.). *Handbook of Tableau Methods.* Kluwer, 1999.
- R. Hähnle. *Tableaux and Related Methods*. In: A. Robinson and A. Voronkov (eds.), *Handbook of Automated Reasoning*, Elsevier Science and MIT Press, 2001.
- Proceedings of the yearly Tableaux conference: http://i12www.ira.uka.d/TABLEAUX/

Tableau - basic definition

Definition

Tableau A tableau is a finite tree with nodes marked with one of the following assertions:

$$w \models \phi \qquad w \not\models \phi \qquad wRw'$$

which is build according to a set of expansion rules (see next slide)

Definition (Branch, open branch and closed branch)

A branch of a tableaux is a sequence $n_1, n_2 \ldots n_k$ where n_1 is the root of the tree, n_k is a leaf, and n_{i+1} is a children of n_i for $1 \leq i < k$.

A closed branch is a branch that contains nodes marked with $w \models \phi$ and $w \not\models \phi$. All other branches are open. If all branches are closed, the tableau is closed.

Tableau Rules for the Propositional Logic

Expansion rules for propositional connectives

$$\frac{w \models \phi \wedge \psi}{\begin{array}{c} w \models \phi \\ w \models \psi \end{array}}$$

$$\frac{w \not\models (\phi \vee \psi)}{\begin{array}{c} w \not\models \phi \\ w \not\models \psi \end{array}}$$

$$\frac{w \models \neg\phi}{w \not\models \phi}$$

$$\frac{w \not\models \neg\phi}{w \models \phi}$$

$$\frac{w \not\models (\phi \supset \psi)}{\begin{array}{c} w \models \phi \\ w \not\models \psi \end{array}}$$

$$\frac{w \models \phi \vee \psi}{w \models \phi \mid w \models \psi}$$

$$\frac{w \not\models (\phi \wedge \psi)}{w \not\models \phi \mid w \not\models \psi}$$

$$\frac{w \models \phi \supset \psi}{w \not\models \phi \mid w \models \psi}$$

Expansion rules for modal operators

$$\frac{w \models \Box\phi}{w' \models \phi}$$ If wRw' is already in the brench

$$\frac{w \not\models \Box\phi}{\begin{array}{c} wRw' \\ w' \not\models \phi \end{array}}$$ wher w' is new in the brench

$$\frac{w \models \Diamond\phi}{\begin{array}{c} wRw' \\ w' \models \phi \end{array}}$$ wher w' is new in the brench

$$\frac{w \not\models \Diamond\phi}{w' \not\models \phi}$$ If wRw' is already in the brench

Applications of expansion rules

- If a branch $\beta = n_1, \ldots, n_k$ contains a node n_i labelled with a premise of one of a rule ρ, and such a rule has not applied yet on this node, then ρ can be applied, and the branch is expanded in the following way
  - if ρ has only one consequence, then β is expanded in $n_1, \ldots n_k, n_{k+1}$ where n_{k+1} is labelled with the consequence of ρ
  - if ρ has two consequences (one on top of the other), then β is expanded in $n_1, \ldots n_k, n_{k+1}, n_{k+2}$ where n_{k+1} and n_{k+2} are labelled with the consequences of ρ
  - if ρ has two alternative consequences (i.e., two consequences separated by a "|"), then β is expanded into two branches $n_1, \ldots n_k, n_{k+2}$ and $n_1, \ldots n_k, n_{k+2}$, where n_{k+1} and n_{k+2} are labelled with the alternative consequences of ρ

Example of tableaux

Example (Check satisfiability of $\Diamond(P \wedge \neg Q) \wedge \Box(P \vee Q)$)

$w \vDash \Diamond(P \wedge \neg Q) \wedge \Box(P \vee Q)$

$w \vDash \Diamond(P \wedge \neg Q)$

$w \vDash \Box(P \vee Q)$

wRw'

$w' \vDash P \wedge \neg Q$

$w' \vDash P$

$w' \vDash \neg Q$

$w' \nvDash Q$

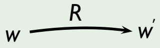
$w' \vDash P \vee Q$

$w' \vDash P$	$w' \vDash Q$
OPEN	CLOSED

- The tableau we have constructed starting from $w \vDash \Diamond(P \wedge \neg Q) \wedge \Box(P \vee Q)$, has an open branch (the one on the left)
- if we collect all the assertions of the form $w \vDash A$ and $w \nvDash A$ for all atomic A and the assertions of the form and wRw', which label the node of such an open branch we obtain

$$wRw', w' \vDash P, \; w' \nvDash Q$$

which corresponds to the model



with A true in w' and B false in w'

Checking validity via tableaux

Example (Check validity of $\Diamond(A \vee B) \equiv \Diamond A \vee \Diamond B$)

To check the validity of $\Diamond(A \vee B) \equiv \Diamond A \vee \Diamond B$, we construct a tableaux that searches for a countermodel. I.e., we check the satisfiability of $\neg(\Diamond(A \vee B) \equiv \Diamond A \vee \Diamond B)$

$w \models \neg(\Diamond(A \vee B) \equiv \Diamond A \vee \Diamond B)$

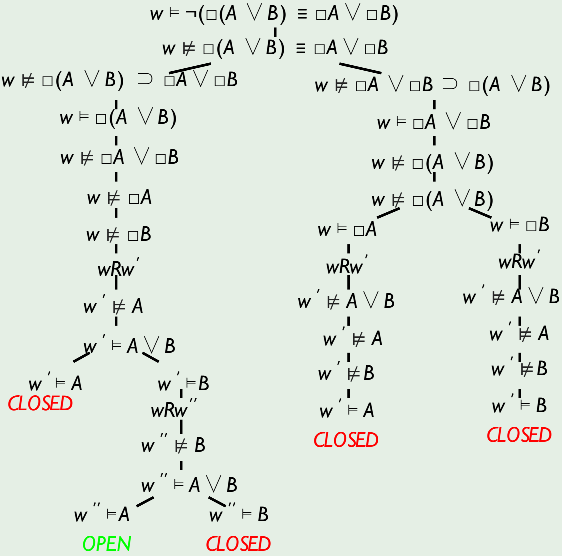
$w \not\models \Diamond(A \vee B) \equiv \Diamond A \vee \Diamond B$

- $w \not\models \Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$
  - $w \models \Diamond(A \vee B)$
  - $w \not\models \Diamond A \vee \Diamond B$
  - $w \not\models \Diamond A$
  - $w \not\models \Diamond B$
  - wRw'
  - $w' \models A \vee B$
    - $w' \models A$
      - $w' \not\models A$
      - CLOSED
    - $w' \models B$
      - $w' \not\models B$
      - CLOSED
- $w \not\models \Diamond A \vee \Diamond B \supset \Diamond(A \vee B)$
  - $w \models \Diamond A \vee \Diamond B$
  - $w \not\models \Diamond(A \vee B)$
    - $w \models \Diamond A$
      - wRw'
      - $w' \models A$
      - $w' \not\models A$
      - CLOSED
    - $w \models \Diamond B$
      - wRw'
      - $w' \models B$ w
      - $' \not\models B$
      - CLOSED

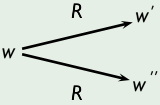
All the branches of the tableaux searching for a model of $\neg(\Diamond(A \vee B) \equiv \Diamond A \vee \Diamond B)$ are closed. This implies that there are no models for such a formulas, i.e., that there are no counter-model for $\Diamond(A \vee B) \equiv \Diamond A \vee \Diamond B$, and finally that $\Diamond(A \vee B) \equiv \Diamond A \vee \Diamond B$, is valid.

Checking validity via tableaux

Example (Check validity of $\Box(A \vee B) \equiv \Box A \vee \Box B$)



The tableau is not closed as there is an open branch. This branch contains the statements: wRw', wRw'', $w' \not\models A$, $w' \models B$ $w'' \models A$ and $w'' \not\models B$, that correspond to the model



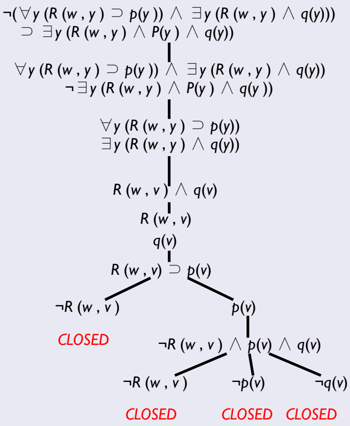
with A false in w', B true in w', A true in w'' and B false in w''.

Comparing Reasoning in ML and FOL

Comparing tableaux reasoning directly in ML and via translation in FOL, we can discover that there are a lot of similarities:

- Reasoning about accessibility relation is explicit in FOL and implicit in ML
- Reasoning about $\forall$ is similar to reasoning about $\Box$
- Reasoning about $\exists$ is similar to reasoning about $\Diamond$

Reasoning in FOL



Reasoning in ML