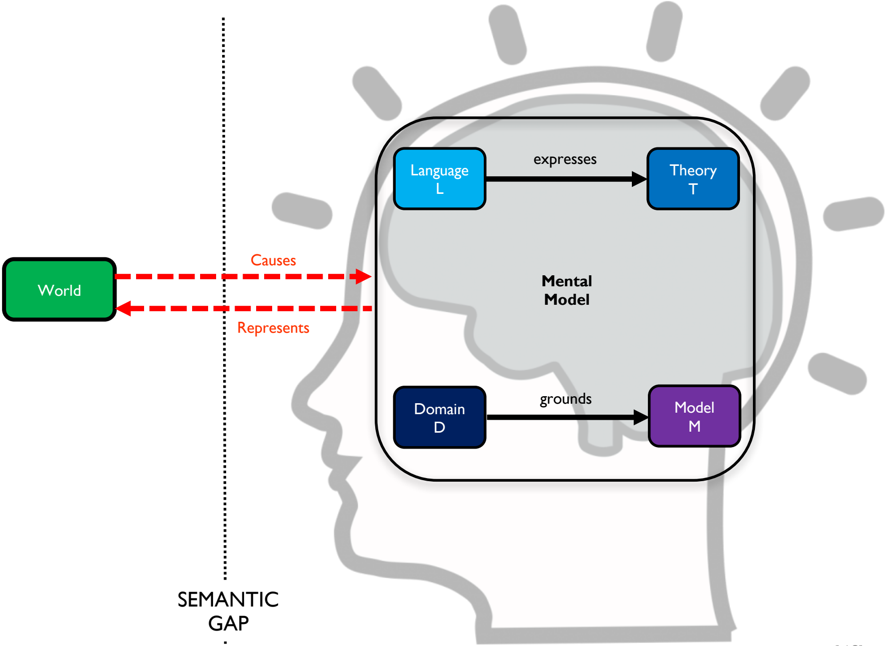
Mathematical Logics
Set Theory*

Fausto Giunchiglia and Mattia Fumagallli

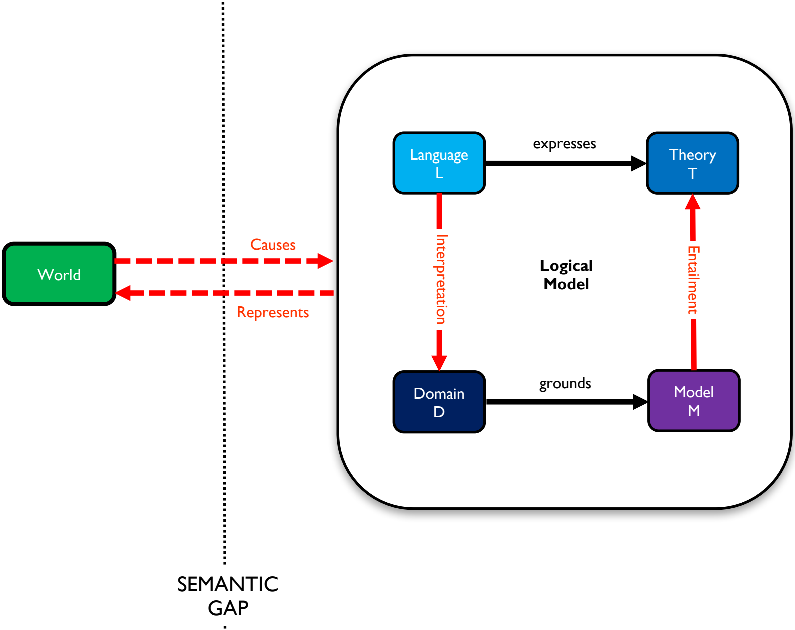
University of Trento

*Originally by Luciano Serafini and Chiara Ghidini
Modified by Fausto Giunchiglia and Mattia Fumagalli

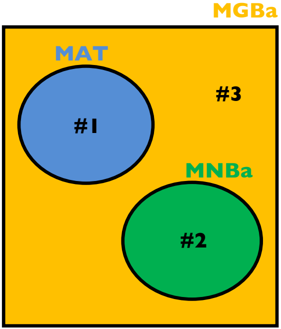
Mental Model



Logical Model



World | Logical Model | Language L | Domain D | Theory T | Model M



L = “*MCT, MAT, MBeBa, MNBa, MCR, MGBa*, $\cap$, $\cup$, $\supset$, …”

T = “*MGBa* $\supset$ *(MAT* $\cup$ *MNBa)*”

D: {#1, #2, #3}

I: “I(MAT) = #1, I(MNBa) = #2, I(MGBa) = #3”

M: “#1 $\in$ MAT, #2 $\in$ MNBa, #3 $\in$ MGBa”

SEMANTIC GAP

Extensional Semantics: Extensions

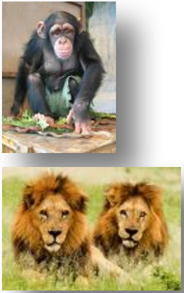
- The meanings which are intended to be attached to the symbols and propositions form the intended interpretation σ (sigma) of the language

- The semantics of a propositional language of classes L are extensional (semantics)

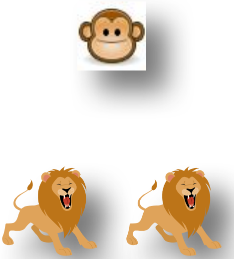
- The extensional semantics of L is based on the notion of "extension" of a formula (proposition) in L

- The extension of a proposition is the totality, or class, or set of all objects D (domain elements) to which the proposition applies

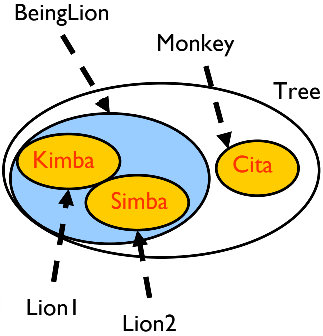
D = {Cita, Kimba, Simba}



The World



The Mental Model



The Formal Model

Class-valuation σ

- In extensional semantics, the first central semantic notion is that of class-valuation (the interpretation function)

  - Given a Class Language L
  - Given a domain of interpretation U

  - A class valuation σ of a propositional language of classes L is a mapping (function) assigning to each formula ψ of L a set σ(ψ) of "objects" (truth-set) in U:

$$\sigma: L \rightarrow pow(U)$$

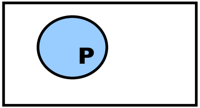
Class-valuation σ

- $\sigma(\bot) = \varnothing$
- $\sigma(\top) = U$ (Universal Class, or Universe)
- $\sigma(P) \subseteq U$, as defined by σ
- $\sigma(\neg P) = \{a \in U \mid a \notin \sigma(P)\} = comp(\sigma(P))$ (Complement)
- $\sigma(P \sqcap Q) = \sigma(P) \cap \sigma(Q)$ (Intersection)
- $\sigma(P \sqcup Q) = \sigma(P) \cup \sigma(Q)$ (Union)

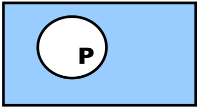
Class-valuation σ

By regarding propositions as classes, it is very convenient to use Venn diagrams

$\sigma(P)$



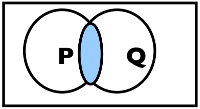
$\sigma(\neg P)$



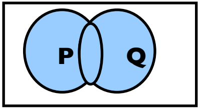
$\sigma(\bot)$

$\sigma(\top)$

$\sigma(P \sqcap Q)$



$\sigma(P \sqcup Q)$



The concept of **set** is considered a primitive concept in math

A set is a collection of elements whose description must be unambiguous and unique: it must be possible to decide whether an element belongs to the set or not.

Examples:

the students in this classroom the points in a straight line the cards in a playing pack

are all sets, while

students that hates math amusing books

are not sets.

Describing Sets

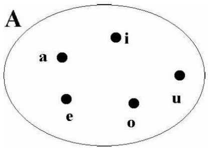
In set theory there are several description methods:

Listing: the set is described listing all its elements Example: $A = \{a, e, i, o, u\}$.

Abstraction: the set is described through a property of its elements

Example: $A = \{x \mid x \text{ is a vowel of the Latin alphabet}\}$

Eulero-Venn Diagrams: graphical representation that supports the formal description



Sets: Basic Concepts

Empty Set: $\emptyset$ is the set containing no elements;
Membership: $a \in A$, element belongs to the set A;

Non membership: $a \notin A$, element a doesn't belong to the set A;

Equality: $A = B$, iff the sets A and B contain the same elements;

inequality: $A \neq B$, iff it is not the case that $A = B$

Subset: $A \subseteq B$, iff all elements in A belong to B too;
Proper subset: $A \subset B$, iff $A \subseteq B$ and $A \neq B$

We define the power set of a set A, denoted with $P(A)$, as the set containing all the subsets of A.

Example: if $A = \{a,b,c\}$, then
$P(A) = \{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\},\}$

If A has n elements, then its power set $P(A)$ contains 2^n elements.

Exercise: prove it!!!

Operations on Sets

Union: given two sets A and B we define the union of A and B as the set containing the elements belonging to A or to B or to both of them, and we denote it with $A \cup B$.

Example: if $A = \{a,b,c\}$, $B = \{a,d,e\}$ then
$A \cup B = \{a,b,c,d,e\}$

Intersection: given two sets A and B we define the intersection of A and B as the set containing the elements that belongs both to A and B, and we denote it with $A \cap B$.

Example: if $A = \{a,b,c\}$, $B = \{a,d,e\}$ then $A \cap B = \{a\}$

Operations on Sets (2)

Difference: given two sets A and B we define the difference of A and B as the set containing all the elements which are members of A, but not members of B, and denote it with $A - B$.

Example: if $A = \{a,b,c\}$, $B = \{a,d,e\}$ then $A - B = \{b,c\}$

Complement: given a universal set U and a set A, where $A \subseteq U$, we define the complement of A in U ,denoted with $\overline{A}$ (or $C_U A$), as the set containing all the elements in U not belonging to A.

Example: if U is the set of natural numbers and A is the set of even numbers (0 included), then the complement of A in U is the set of odd numbers.

Examples:

Given $A = \{a, e, i, o, \{u\}\}$ and $B = \{i, o, u\}$, consider the following statements:

1. $B \in A$ NO!
2. $(B - \{i, o\}) \in A$ OK
3. $\{a\} \cup \{i\} \subset A$ OK
4. $\{u\} \subset A$ NO!
5. $\{\{u\}\} \subset A$ OK
6. $B - A = \emptyset$ NO! $B - A = \{u\}$
7. $i \in A \cap B$ OK
8. $\{i, o\} = A \cap B$ OK

Sets: Operation Properties

$$A \cap A = A,$$
$$A \cup A = A$$

$$A \cap B = B \cap A,$$
$$A \cup B = B \cup A \text{ (commutative)}$$

$$A \cap \varnothing = \varnothing$$
$$A \cup \varnothing = A$$

$$(A \cap B) \cap C = A \cap (B \cap C),$$
$$(A \cup B) \cup C = A \cup (B \cup C) \text{ (associative)}$$

Sets: Operation Properties (2)

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  (*distributive*)
- $\overline{A \cap B} = \overline{A} \cup \overline{B}$
  $\overline{A \cup B} = \overline{A} \cap \overline{B}$ (De Morgan laws)
- **Exercise**: Prove the validity of all the properties.

Cartesian Product

Given two sets A and B, we define the Cartesian product of A and B as the set of ordered couples (a, b) where $a \in A$ and $b \in B$; formally,
$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

Notice that: $A \times B \neq B \times A$

Cartesian Product (2)

- **Examples**:
  - given $A = \{1, 2, 3\}$ and $B = \{a, b\}$, then
    $A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$ and
    $B \times A = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$.
  - Cartesian coordinates of the points in a plane are an example of the Cartesian product R × R
- The Cartesian product can be computed on any number n of sets $A_1, A_2, \ldots, A_n$, $A_1 \times A_2 \times \ldots \times A_n$ is the set of ordered n-tuple $(x_1, \ldots, x_n)$ where $x_i \in A_i$ for each $i = 1 \ldots n$.

Relations

- A relation R from the set A to the set B is a subset of the Cartesian product of A and B: $R \subseteq A \times B$; if $(x,y) \in R$, then we will write xRy for 'x is R-related to y'.

- A binary relation on a set A is a subset $R \subseteq A \times A$

- **Examples**:
  - given $A = \{1, 2, 3, 4\}$, $B = \{a, b, d, e, r, t\}$ and aRb iff in the Italian name of a there is the letter b, then $R = \{(2, d), (2, e), (3, e), (3, r), (3, t), (4, a), (4, r), (4, t)\}$
  - given $A = \{3, 5, 7\}$, $B = \{2, 4, 6, 8, 10, 12\}$ and aRb iff a is a divisor of b, then $R = \{(3, 6), (3, 12), (5, 10)\}$

- **Exercise**: in prev. example, let aRb iff $a + b$ is an even number
  $R = ?$

Relations (2)

- Given a relation R from A to B,
  - the domain of R is the set $Dom(R) = \{a \in A \mid \text{there exists a } b \in B, aRb\}$
  - the co-domain of R is the set $Cod(R) = \{b \in B \mid \text{there exists an } a \in A, aRb\}$

- Let R be a relation from A to B. The inverse relation of R is the relation $R^{-1} \subseteq B \times A$ where $R^{-1} = \{(b, a) \mid (a, b) \in R\}$

Relations Properties

- Let R be a binary relation on A. R is
  - reflexive iff aRa for all $a \in A$;
  - symmetric iff aRb implies bRa for all $a, b \in A$;
  - transitive iff aRb and bRc imply aRc for all $a, b, c \in A$;
  - anti-symmetric iff aRb and bRa imply $a = b$ for all $a, b \in A$;

Equivalence

- Let R be a binary relation on a set A. R is an equivalence relation iff it satisfies all the following properties:
  - reflexive
  - symmetric
  - transitive
- an equivalence relation is usually denoted with $\sim$ or $\equiv$

Set Partition

Let A be a set, a partition of A is a family F of non-empty subsets of A s.t.:

- the subsets are pairwise disjoint
- the union of all the subsets is the set A

Notice that: each element of A belongs to exactly one subset in F.

Equivalence Class

- Let A be a set and $\equiv$ an equivalence relation on A, given an $x \in A$ we define **equivalence class** X the set of elements $x' \in A$ s.t. $x' \equiv x$, formally
  $X = \{x' \mid x' \equiv x\}$
- Notice that: any element x is sufficient to obtain the equivalence class X , which is denoted also with $[x]$
  - $x \equiv x'$ implies $[x] = [x'] = X$
- We define **quotient set** of A with respect to an equivalence relation $\equiv$ as the set of equivalence classes defined by $\equiv$ on A, and denote it with $A/\equiv$

Equivalence Class (2)

- Theorem: Given an equivalence relation $\equiv$ on A, the equivalence classes defined by $\equiv$ on A are a partition of A. Similarly, given a partition on A, the relation R defined as xRx' iff *x and x' belong to the same subset*, is an equivalence relation on A.

Equivalence Class (3)

- **Example**: Parallelism relation.
  Two straight lines in a plane are parallel if they do not have any point in common or if they coincide.
- The parallelism relation $\parallel$ is an equivalence relation since it is:
  - reflexive $r \parallel r$
  - symmetric $r \parallel s$ implies $s \parallel r$
  - transitive $r \parallel s$ and $s \parallel t$ imply $r \parallel t$
- We can thus obtain a partition in equivalence classes: intuitively, each class represent a direction in the plane.

Order Relation

- Let A be a set and R be a binary relation on A. R is an order (partial), usually denoted with $\leq$, if it satisfies the following properties:
  - reflexive $a \leq a$
  - anti-symmetric $a \leq b$ and $b \leq a$ imply $a = b$
  - transitive $a \leq b$ and $b \leq c$ imply $a \leq c$
- If the relation holds for all $a, b \in A$ then it is a total order
- A relation is a strict order, denoted with $<$, if it satisfies the following properties:
  - transitive $a < b$ and $b < c$ imply $a < c$
  - for all $a, b \in A$ either $a < b$ or $b < a$ or $a = b$

Functions

Given two sets A and B, a function f from A to B is a relation that associates to each element a in A exactly one element b in B. Denoted with

$f : A \to B$

The domain of f is the whole set A; the image of each element a in A is the element b in B s.t. $b = f(a)$; the co-domain of f (or image of f) is a subset of B defined as follows:

$Im_f = \{b \in B \mid \text{there exists an } a \in A \text{ s.t. } b = f(a)\}$

Notice that: it can be the case that the same element in B is the image of several elements in A.

Classes of Functions

A function $f : A \to B$ is surjective if each element in B is image of some elements in A: for each $b \in B$ there exists an $a \in A$ s.t. $f(a) = b$

A function $f : A \to B$ is injective if distinct elements in A have distinct images in B:
for each $b \in Im_f$ there exists a unique $a \in A$ s.t. $f(a) = b$

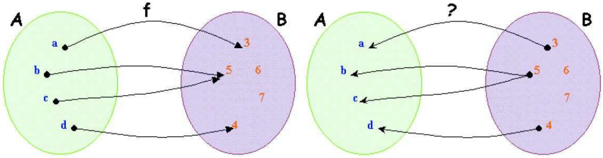
A function $f : A \to B$ is bijective if it is injective and surjective:
for each $b \in B$ there exists a unique $a \in A$ s.t. $f(a) = b$

If $f : A \to B$ is bijective we can define its inverse function:

$$f^{-1} : B \to A$$

For each function f we can define its inverse relation; such a relation is a function iff f is bijective.

Example:



the inverse relation of f is NOT a function.

Composed Function

Let $f:A \to B$ and $g:B \to C$ be functions. The composition of f and g is the function $g \circ f:A \to C$ obtained by applying f and then g:

$$(g \circ f)(a) = g(f(a)) \text{ for each } a \in A$$
$$g \circ f = \{(a, g(f(a)) \mid a \in A)\}$$