Mathematical Logic - 2015

Propositional Logic: exercises

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Expansion Rules of Propositional Tableau

α rules

$$\frac{\phi \wedge \psi}{\begin{array}{c}\phi \\ \psi\end{array}} \qquad \frac{\neg(\phi \vee \psi)}{\begin{array}{c}\neg\phi \\ \neg\psi\end{array}} \qquad \frac{\neg(\phi \supset \psi)}{\begin{array}{c}\phi \\ \neg\psi\end{array}}$$

$\neg\neg$-Elimination

$$\frac{\neg\neg\phi}{\phi}$$

β rules

$$\frac{\phi \vee \psi}{\phi \mid \psi} \qquad \frac{\neg(\phi \wedge \psi)}{\neg\phi \mid \neg\psi} \qquad \frac{\phi \supset \psi}{\neg\phi \mid \psi}$$

Branch Closure

$$\frac{\begin{array}{c}\phi \\ \neg\phi\end{array}}{X}$$

Tableaux: exercise 1

Is the the following formula satisfiable?

$$\neg((P \wedge Q) \rightarrow \neg\neg P)$$

$$(P \wedge Q)$$

$$\neg\neg\neg P$$

$$\neg P$$

$$P$$

$$Q$$

X

Tableaux: Exercise 2

Is the following formula valid? (and satisifiable?)

$((P \lor Q) \land \neg P)$

$(P \lor Q)$

$\neg P$

P $\qquad\qquad$ Q

X

Tableaux: Exercise 3

Is the following formula satisifable?

$\neg((P \rightarrow Q) \wedge (P \wedge Q \rightarrow R) \rightarrow (P \rightarrow R))$

$(P \rightarrow Q) \wedge (P \wedge Q \rightarrow R)$

$\neg(P \rightarrow R)$

$(P \rightarrow Q)$

$(P \wedge Q \rightarrow R)$

P

$\neg R$

$\neg P$ | Q

X | $\neg(P \wedge Q)$ | R

$\neg P$ | $\neg Q$ | X

X | X

Tableaux: Exercise 4

Is the following argumentation valid?

$$P \vee Q, P \vee \neg Q \vdash \neg\neg P$$

$P \vee Q$

$P \vee \neg Q$

$\neg\neg\neg P$

$\neg P$

P Q

X

P $\neg Q$

X   X

Tableaux: Exercise 5

Check the model for the following formula:

$$\neg(P \vee Q \rightarrow P \wedge Q)$$

$(P \vee Q)$

$\neg(P \wedge Q)$

P		Q	
$\neg P$	$\neg Q$	$\neg P$	$\neg Q$
X			X

I(P) = True; I(Q) = False
I(P) = False; I(Q) = True

Tableaux: Exercise 6

Is the following argumentation valid?

$$P, (P \rightarrow Q) \vdash P \leftrightarrow Q$$

P

$P \rightarrow Q$

$\neg(P \leftrightarrow Q)$

$\neg P$ $\qquad$ Q

X

P $\qquad$ Q

$\neg Q$ $\qquad$ $\neg P$

X $\qquad$ X

Reduction in CNF

$$
\begin{aligned}
CNF(p) &= p \quad \text{if } p \in \mathcal{P} \\
CNF(\neg p) &= \neg p \quad \text{if } p \in \mathcal{P} \\
CNF(\phi \to \psi) &= CNF(\neg\phi) \otimes CNF(\psi) \\
CNF(\phi \wedge \psi) &= CNF(\phi) \wedge CNF(\psi) \\
CNF(\phi \vee \psi) &= CNF(\phi) \otimes CNF(\psi) \\
CNF(\phi \equiv \psi) &= CNF(\phi \to \psi) \wedge CNF(\psi \to \phi) \\
CNF(\neg\neg\phi) &= CNF(\phi) \\
CNF(\neg(\phi \to \psi)) &= CNF(\phi) \wedge CNF(\neg\psi) \\
CNF(\neg(\phi \wedge \psi)) &= CNF(\neg\phi) \otimes CNF(\neg\psi) \\
CNF(\neg(\phi \vee \psi)) &= CNF(\neg\phi) \wedge CNF(\neg\psi) \\
CNF(\neg(\phi \equiv \psi)) &= CNF(\phi \wedge \neg\psi) \otimes CNF(\psi \wedge \neg\phi)
\end{aligned}
$$

Convert a formula in CNF

- $\neg(\neg p \vee q) \vee (r \rightarrow \neg s)$

$\neg(\neg p \vee q) \vee (\neg r \vee \neg s)$

$(\neg\neg p \wedge \neg q) \vee (\neg r \vee \neg s)$

$(p \wedge \neg q) \vee (\neg r \vee \neg s)$ NNF

$(p \vee \neg r \vee \neg s) \wedge (\neg q \vee \neg r \vee \neg s)$

- $(C \rightarrow \neg A) \wedge \neg(B \wedge \neg A)$

  $(\neg C \vee \neg A) \wedge \neg(B \wedge \neg A)$
  $(\neg C \vee \neg A) \wedge (\neg B \vee \neg\neg A)$
  $(\neg C \vee \neg A) \wedge (\neg B \vee \neg A)$

Convert a formula in CNF

- $(A \wedge C) \rightarrow (C \rightarrow \neg A)$

  $\neg(A \wedge C) \vee (C \rightarrow \neg A)$
  $\neg(A \wedge C) \vee (\neg C \vee \neg A)$
  $\neg A \vee \neg C \vee \neg C \vee \neg A$
  $\neg A \vee \neg C$

- $((A \rightarrow B) \rightarrow A) \rightarrow A$

  $\neg(\neg(\neg A \vee B) \vee A) \vee A$
  $\neg((A \wedge \neg B) \vee A) \vee A$
  $\neg((A \vee A) \wedge (A \vee \neg B)) \vee A$
  $\neg A \vee \neg(A \vee \neg B) \vee A$
  $\neg(A \vee \neg B)$
  $\neg A \wedge B$