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Mathematical Logic

First Order Logic and Propositinal Logic

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Finite domain

If we are interested in representing facts on a finite domain that contains *n* elements we can use the following theorem:

Theorem

The formula

$$\phi_{|\Delta|=n} = \exists x_1, \ldots, x_n \left(\bigwedge_{i \neq j=1}^{n} x_i \neq x_j \wedge \forall x \left(\bigvee_{i=1}^{n} x_i = x \right) \right)$$

is true in $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ *if and only if* $|\Delta^{\mathcal{I}}| = n$, *i.e., the cardinality of* Δ *is equal to n, i.e.,* $\Delta^{\mathcal{I}}$ *contains exactly n elements.*

Finite domain

Proof.

We show that if then

1. If then there are $d_1, \dots, d_n \in \Delta^{\mathcal{I}}$ s.t.
2. $\mathcal{I} \models \bigwedge_{i \neq j=1}^{n} x_i \neq x_j \wedge \forall x \left(\bigvee_{i=1}^{n} x_i = x \right) [a[x_1 := d_1, \dots, x_n := d_n]]$
3. $\mathcal{I} \models \bigwedge_{i \neq j=1}^{n} x_i \neq x_j [a[x_1 := d_1, \dots, x_n := d_n]$
4. From 3 we have that for all $1 \leq i \neq j \leq n$,
   $\mathcal{I} \models x_i \neq x_j [a[x_i := d_i, x_j = d_j]]$
5. this implies that $d_i \neq d_j$ for $1 \leq i \neq j \leq n$.
6. since $d_1, \dots, d_n \in \Delta^{\mathcal{I}}$, we have that .
7. from 2 we have $\mathcal{I} \models \forall x \left(\bigvee_{i=1}^{n} x_i = x \right) [a[x_1 := d_1, \dots, x_n := d_n]]$
8. ths implies that for any $d \in \Delta^{\mathcal{I}}$,
   $\mathcal{I} \models \bigvee_{i=1}^{n} x_i = x [a[x_1 := d_1, \dots, x_n := d_n, x := d]]$
9. which implies that for some i, $I \models x_i = x[a[x_i := d_i, x = d]]$, i.e., $d_i = d$ for some $1 \leq i \leq n$.
10. Since this is true for all $d \in \Delta^{\mathcal{I}}$, then .

Finite domain, with names for every element

Unique Name Assumption (UNA)

Is the assumption under which the language contains a name for each element of the domain, i.e., the language contains the constant $c_1, \ldots, c_n$, and each constant is the name of one and only one domain element.

Theorem

The formula

$$\phi_{\Delta=\{c_1,\ldots,c_n\}} = \bigwedge_{i\neq j=1}^{n} c_i \neq c_j \wedge \forall x \left(\bigvee_{i=1}^{n} c_i = x \right)$$

$\phi_{\Delta=\{c_1,\ldots,c_n\}}$ *is also called Unique Name Assumption.*

Proof.

The proof is similar to the one of the previous theorem. Try it by exercise.

Finite domain - Grounding

Under the hypothesis of finite domain with a constant name for every elements, First order formulas can be propositionalized, aka grounded as follows:

$$\phi_{\Delta=\{c_1,\dots,c_n\}} \models \forall x \phi(x) \equiv \phi(c_1) \wedge \cdots \wedge \phi(c_n) \quad (1)$$

$$\phi_{\Delta=\{c_1,\dots,c_n\}} \models \exists x \phi(x) \equiv \phi(c_1) \vee \cdots \vee \phi(c_n) \quad (2)$$

Generalizing:

$$\phi_{\Delta=\{c_1,\dots,c_n\}} \models \forall x_1 \dots x_k \phi(x_1,\dots,x_k) \equiv \bigwedge_{\substack{c_{i_1},\dots,c_{i_k} \in \\ \{c_1,\dots,c_n\}}} \phi(c_{i_1},\dots,c_{i_k}) \quad (3)$$

$$\phi_{\Delta=\{c_1,\dots,c_n\}} \models \exists x_1 \dots x_k \phi(x_1,\dots,x_k) \equiv \bigvee_{\substack{c_{i_1},\dots,c_{i_k} \in \\ \{c_1,\dots,c_n\}}} \phi(c_{i_1},\dots,c_{i_k}) \quad (4)$$

Finite predicate extension

The assumption that states that a predicate P is true only for a finite set of objects for which the language contains a name, can be formalized by the following formulas:

$$\forall x(P(x) \equiv x = c_1 \vee \cdots \vee x = c_n)$$

Example

- The days of the week are: Monday, Tuesday, ..., Sunday.

$$\forall x(\text{WeekDay}(x) \equiv x = \text{Mon} \vee x = \text{Tue} \vee \cdots \vee x = \text{Sun})$$

- The WorkingDays Monday, Tuesday, ..., Friday:

$$\forall x(\text{WorkingDay}(x) \equiv x = \text{Mon} \vee x = \text{Tue} \vee \cdots \vee x = \text{Fri})$$

Infinite domain

Is it possible to write a (set of) formula(s) that are satisfied only by an interpretation with infinite domain

Theorem

Let $\phi_{\textit{inf-dom}}$ *be the formula:*

$$\begin{aligned}\phi_{\textit{inf-dom}} \quad = \quad & \forall x \neg R(x,x) \ \wedge \\ & \forall x \forall y \forall z (R(x,y) \wedge R(y,z) \supset R(x,z)) \ \wedge \\ & \forall x \exists y R(x,y)\end{aligned}$$

If $\mathcal{I} \models \phi_{\textit{inf-dom}}$ *then* $|\Delta^{\mathcal{I}}| = \infty$.

Observe that:

- $\forall x \forall y \forall z (R(x,y) \wedge R(y,z) \supset R(x,z))$ represents the fact that R is interpreted in a transitive relation
- $\forall x \neg R(x,x)$ represents the fact that R is anti-reflexive

Infinite domain

Proof.

- By definition there is a $d_0 \in \Delta^{\mathcal{I}}$. Since $\mathcal{I} \models \forall x \exists y R(x, y)$, there must be a $d_1 \in \Delta^{\mathcal{I}}$ such that $\langle d_0, d_1 \rangle \in R^{\mathcal{I}}$. For the same reason there must be a $d_2 \in \Delta^{\mathcal{I}}$, such that $\langle d_1, d_2 \rangle \in R^{\mathcal{I}}$. And so on .... This means that there must be an infinite sequence $d_0, d_1, d_2, \dots$ such that $\langle d_i, d_{i+1} \rangle$, for every $i \geq 0$.
- Since $\mathcal{I} \models \forall x \forall y \forall z (R(x, y) \wedge R(y, z) \supset R(x, z))$, then for all $i < j$, $\langle d_i, d_j \rangle \in R^{\mathcal{I}}$.
- suppose, by contradiction, that $|\Delta^{\mathcal{I}}| = k$ for some finite number k. This means there is an i, j with $0 \leq i < j \leq k + 1$ such that $d_i = d_j$.
- The fact that $\langle d_i, d_j \rangle \in R^{\mathcal{I}}$ implies that $\langle d_i, d_i \rangle \in R^{\mathcal{I}}$. But this contradicts the fact that $\mathcal{I} \models \forall x \neg R(x, x)$.

$\square$