Mathematical Logic

Practical Class: Set Theory

Chiara Ghidini

FBK-IRST, Trento, Italy

2014/2015

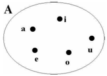
Sets: Basic Concepts

- The concept of **set** is considered a primitive concept in math
- A set is a collection of elements whose description must be unambiguous and unique: it must be possible to decide whether an element belongs to the set or not.
- **Examples:**
  - *the students in this classroom*
  - *the points in a straight line*
  - *the cards in a playing pack*
- are all sets, while
  - *students that hates math*
  - *amusing books*

  are not sets.

Describing Sets

- In set theory there are several description methods:
  - Listing: the set is described listing all its elements
    Example: $A = \{a, e, i, o, u\}$.
  - Abstraction: the set is described through a property of its elements
    Example: $A = \{x \mid x \text{ is a vowel of the Latin alphabet }\}$.
  - Eulero-Venn Diagrams: graphical representation that supports the formal description



Sets: Basic Concepts (2)

- Empty Set: $\emptyset$, is the set containing no elements;
- Membership: $a \in A$, element a belongs to the set A;
  - *Non membership*: $a \notin A$, element a doesn't belong to the set A;
- Equality: $A = B$, iff the sets A and B contain the same elements;
  - *inequality*: $A \neq B$, iff it is not the case that $A = B$;
- Subset: $A \subseteq B$, iff all elements in A belong to B too;
- Proper subset: $A \subset B$, iff $A \subseteq B$ and $A \neq B$.

Outline
Set Theory
Relations
Functions
Basic Concepts
Operations on Sets
Operation Properties

Power set

- We define the power set of a set A, denoted with $P(A)$, as the set containing all the subsets of A.
- **Example**: if $A = \{a, b, c\}$, then $P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \}$
- If A has n elements, then its power set $P(A)$ contains 2^n elements.
  - Exercise: prove it!!!

Outline
Set Theory
Relations
Functions
Basic Concepts
Operations on Sets
Operation Properties

Operations on Sets

- Union: given two sets A and B we define the union of A and B as the set containing the elements belonging to A or to B or to both of them, and we denote it with $A \cup B$.
  - **Example**: if $A = \{a, b, c\}$, $B = \{a, d, e\}$ then $A \cup B = \{a, b, c, d, e\}$
- Intersection: given two sets A and B we define the intersection of A and B as the set containing the elements that belongs both to A and B, and we denote it with $A \cap B$.
  - **Example**: if $A = \{a, b, c\}$, $B = \{a, d, e\}$ then $A \cap B = \{a\}$

Outline
Set Theory
Relations
Functions
Basic Concepts
Operations on Sets
Operation Properties

Operations on Sets (2)

- Difference: given two sets A and B we define the difference of A and B as the set containing all the elements which are members of A, but not members of B, and denote it with $A - B$.
  - **Example**: if $A = \{a, b, c\}$, $B = \{a, d, e\}$ then $A - B = \{b, c\}$
- Complement: given a universal set U and a set A, where $A \subseteq U$, we define the complement of A in U ,denoted with $\overline{A}$ (or $C_U A$), as the set containing all the elements in U not belonging to A.
  - **Example**: if U is the set of natural numbers and A is the set of even numbers (0 included), then the complement of A in U is the set of odd numbers.

Sets: Examples

- **Examples**:
  - Given $A = \{a, e, i, o, \{u\}\}$ and $B = \{i, o, u\}$, consider the following statements:
    1. $B \in A$ NO!
    2. $(B - \{i, o\}) \in A$ OK
    3. $\{a\} \cup \{i\} \subset A$ OK
    4. $\{u\} \subset A$ NO!
    5. $\{\{u\}\} \subset A$ OK
    6. $B - A = \emptyset$ NO! $B - A = \{u\}$
    7. $i \in A \cap B$ OK
    8. $\{i, o\} = A \cap B$ OK

Outline
Set Theory
Relations
Functions
Basic Concepts
Operations on Sets
Operation Properties

Sets: Exercises

- **Exercises**:
  - Given $A = \{t, z\}$ and $B = \{v, z, t\}$ consider the following statements:
    1. $A \in B$
    2. $A \subset B$
    3. $z \in A \cap B$
    4. $v \subset B$
    5. $\{v\} \subset B$
    6. $v \in A - B$
  - Given $A = \{a, b, c, d\}$ and $B = \{c, d, f\}$
    - find a set X s.t. $A \cup B = B \cup X$; is this set unique?
    - there exists a set Y s.t. $A \cup Y = B$?

Outline
Set Theory
Relations
Functions
Basic Concepts
Operations on Sets
Operation Properties

Sets: Exercises (2)

- **Exercises**:
  - Given $A = \{0, 2, 4, 6, 8, 10\}$, $B = \{0, 1, 2, 3, 4, 5, 6\}$ and $C = \{4, 5, 6, 7, 8, 9, 10\}$, compute:
    - $A \cap B \cap C$, $A \cup (B \cap C)$, $A - (B - C)$
    - $(A \cup B) \cap C$, $(A - B) - C$, $A \cap (B - C)$
  - Describe 3 sets A, B, C s.t. $A \cap (B \cup C) \neq (A \cap B) \cup C$

Outline
Set Theory
Relations
Functions
Basic Concepts
Operations on Sets
Operation Properties

Sets: Operation Properties

- $A \cap A = A$,
  $A \cup A = A$
- $A \cap B = B \cap A$,
  $A \cup B = B \cup A$ (*commutative*)
- $A \cap \emptyset = \emptyset$,
  $A \cup \emptyset = A$
- $(A \cap B) \cap C = A \cap (B \cap C)$,
  $(A \cup B) \cup C = A \cup (B \cup C)$ (*associative*)

Sets: Operation Properties(2)

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  (*distributive*)
- $\overline{A \cap B} = \overline{A} \cup \overline{B}$,
  $\overline{A \cup B} = \overline{A} \cap \overline{B}$ (*De Morgan laws*)
- **Exercise**: Prove the validity of all the properties.

Cartesian Product

- Given two sets A and B, we define the Cartesian product of A and B as the set of ordered couples (a, b) where $a \in A$ and $b \in B$; formally,
  $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$
- Notice that: $A \times B \neq B \times A$

Cartesian Product (2)

- **Examples**:
  - given $A = \{1, 2, 3\}$ and $B = \{a, b\}$, then
    $A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$ and
    $B \times A = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$.
  - Cartesian coordinates of the points in a plane are an example of the Cartesian product $\Re \times \Re$
- The Cartesian product can be computed on any number n of sets $A_1, A_2 \ldots, A_n$, $A_1 \times A_2 \times \ldots \times A_n$ is the set of ordered n-tuple $(x_1, \ldots, x_n)$ where $x_i \in A_i$ for each $i = 1 \ldots n$.

Relations

- A relation R from the set A to the set B is a subset of the Cartesian product of A and B: $R \subseteq A \times B$; if $(x, y) \in R$, then we will write xRy for 'x is R-related to y'.
- A binary relation on a set A is a subset $R \subseteq A \times A$
- **Examples**:
  - given $A = \{1, 2, 3, 4\}$, $B = \{a, b, d, e, r, t\}$ and aRb iff in the Italian name of a there is the letter b, then
    $R = \{(2, d), (2, e), (3, e), (3, r), (3, t), (4, a), (4, r), (4, t)\}$
  - given $A = \{3, 5, 7\}$, $B = \{2, 4, 6, 8, 10, 12\}$ and aRb iff a is a divisor of b, then
    $R = \{(3, 6), (3, 12), (5, 10)\}$
- **Exercise**: in prev example, let aRb iff $a + b$ is an even number
  $R = ?$

Relations (2)

- Given a relation R from A to B,
  - the domain of R is the set $Dom(R) = \{a \in A \mid \text{there exists a } b \in B, aRb\}$
  - the co-domain of R is the set $Cod(R) = \{b \in B \mid \text{there exists an } a \in A, aRb\}$
- Let R be a relation from A to B. The inverse relation of R is the relation $R^{-1} \subseteq B \times A$ where $R^{-1} = \{(b, a) \mid (a, b) \in R\}$

Relation properties

- Let R be a binary relation on A. R is
  - reflexive iff aRa for all $a \in A$;
  - symmetric iff aRb implies bRa for all $a, b \in A$;
  - transitive iff aRb and bRc imply aRc for all $a, b, c \in A$;
  - anti-symmetric iff aRb and bRa imply $a = b$ for all $a, b \in A$;

Equivalence Relation

- Let R be a binary relation on a set A. R is an equivalence relation iff it satisfies all the following properties:
  - reflexive
  - symmetric
  - transitive
- an equivalence relation is usually denoted with $\sim$ or $\equiv$

Set Partition

- Let A be a set, a partition of A is a family F of non-empty subsets of A s.t.:
  - the subsets are pairwise disjoint
  - the union of all the subsets is the set A
- Notice that: each element of A belongs to exactly one subset in F.

Equivalence Classes

- Let A be a set and $\equiv$ an equivalence relation on A, given an $x \in A$ we define equivalence class X the set of elements $x' \in A$ s.t. $x' \equiv x$, formally $X = \{x' \mid x' \equiv x\}$
- Notice that: any element x is sufficient to obtain the equivalence class X, which is denoted also with $[x]$
  - $x \equiv x'$ implies $[x] = [x'] = X$
- We define quotient set of A with respect to an equivalence relation $\equiv$ as the set of equivalence classes defined by $\equiv$ on A, and denote it with $A/\equiv$

Equivalence Classes (2)

- Theorem: Given an equivalence relation $\equiv$ on A, the equivalence classes defined by $\equiv$ on A are a partition of A. Similarly, given a partition on A, the relation R defined as xRx' iff *x and x' belong to the same subset*, is an equivalence relation on A.

Equivalence classes (3)

- **Example**: Parallelism relation.
  Two straight lines in a plane are parallel if they do not have any point in common or if they coincide.
- The parallelism relation $||$ is an equivalence relation since it is:
  - reflexive $r||r$
  - symmetric $r||s$ implies $s||r$
  - transitive $r||s$ and $s||t$ imply $r||t$
- We can thus obtain a partition in equivalence classes: intuitively, each class represent a direction in the plane.

Order Relation

- Let A be a set and R be a binary relation on A. R is an order (partial) , usually denoted with $\leq$, if it satisfies the following properties:
  - reflexive $a \leq a$
  - anti-symmetric $a \leq b$ and $b \leq a$ imply $a = b$
  - transitive $a \leq b$ and $b \leq c$ imply $a \leq c$
- If the relation holds for all $a, b \in A$ then it is a total order
- A relation is a strict order, denoted with $<$, if it satisfies the following properties:
  - transitive $a < b$ and $b < c$ imply $a < c$
  - for all $a, b \in A$ either $a < b$ or $b < a$ or $a = b$

Relations : Exercises

- **Exercises**:
  - Decide whether the following relations $R : \mathbb{Z} \times \mathbb{Z}$ are symmetric, reflexive and transitive:
    - $R = \{(n, m) \in \mathbb{Z} \times \mathbb{Z} : n = m\}$
    - $R = \{(n, m) \in \mathbb{Z} \times \mathbb{Z} : |n - m| = 5\}$
    - $R = \{(n, m) \in \mathbb{Z} \times \mathbb{Z} : n \geq m\}$

Relations : Exercises (2)

- **Exercises**:
  - Let $X = \{1, 2, 3, \ldots, 30, 31\}$. Consider the relation on X : *xRy if the dates x and y of January 2006 are on the same day of the week (Monday, Tuesday ..)*. Is R an equivalence relation? If this is the case describe its equivalence classes.
  - Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
    - Consider the following relation on X: *xRy iff $x + y$ is an even number*. Is R an equivalence relation? If this is the case describe its equivalence classes.
    - Consider the following relation on X: *xRy iff $x + y$ is an odd number*. Is R an equivalence relation? If this is the case describe its equivalence classes.

Relations : Exercises (3)

- **Exercises**:
  - Let X be the set of straight-lines in the plane, and let x be a point in the plane. Are the following relations equivalence relations? If this is the case describe the equivalence classes.
    - $r \sim s$ iff *r and s are parallel*
    - $r \sim s$ iff *the distance between r and x is equal to the distance between s and x*
    - $r \sim s$ iff *r and s are perpendicular*
    - $r \sim s$ iff *the distance between r and x is greater or equal to the distance between s and x*
    - $r \sim s$ iff *both r and s pass through x*

Relations : Exercises (4)

- **Exercises**:
  - Let *div* be a relation on $\mathbb{N}$ defined as *a div b iff a divides b*. Where *a* divides *b* iff there exists an $n \in \mathbb{N}$ s.t. $a * n = b$
    - Is *div* an equivalence relation?
    - Is *div* an order?

Functions

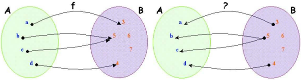
- Given two sets A and B, a function f from A to B is a relation that associates to each element a in A exactly one element b in B. Denoted with
  $f : A \longrightarrow B$
- The domain of f is the whole set A; the image of each element a in A is the element b in B s.t. $b = f(a)$; the co-domain of f (or image of f) is a subset of B defined as follows:
  $Im_f = \{b \in B \mid \text{there exists an } a \in A \text{ s.t. } b = f(a)\}$
- Notice that: it can be the case that the same element in B is the image of several elements in A.

Classes of functions

- A function $f : A \longrightarrow B$ is surjective if each element in B is image of some elements in A:
  for each $b \in B$ there exists an $a \in A$ s.t. $f(a) = b$
- A function $f : A \longrightarrow B$ is injective if distinct elements in A have distinct images in B:
  for each $b \in Im_f$ there exists a unique $a \in A$ s.t. $f(a) = b$
- A function $f : A \longrightarrow B$ is bijective if it is injective and surjective:
  for each $b \in B$ there exists a unique $a \in A$ s.t. $f(a) = b$

Inverse Function

- If $f : A \longrightarrow B$ is bijective we can define its inverse function:
  $f^{-1} : B \longrightarrow A$
- For each function f we can define its inverse relation; such a relation is a function iff f is bijective.
- Example:



the inverse relation of f is NOT a function.

Composed functions

- Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be functions. The composition of f and g is the function $g \circ f : A \longrightarrow C$ obtained by applying f and then g:
  $(g \circ f)(a) = g(f(a))$ for each $a \in A$
  $g \circ f = \{(a, g(f(a)) \mid a \in A)\}$

Functions : Exercises

- **Exercises**:
  - Given $A = \{$ *students that passed the Logic exam* $\}$ and $B = \{18, 19, .., 29, 30, 30L\}$, and let $f : A \longrightarrow B$ be the function defined as $f(x) =$ *grade of* x *in Logic*. Answer the following questions:
    - What is the image of f?
    - Is f bijective?
  - Let A be the set of all people, and let $f : A \longrightarrow A$ be the function defined as $f(x) =$ *father of* x. Answer the following questions:
    - What is the image of f?
    - Is f bijective?
    - Is f invertible?
  - Let $f : \mathbb{N} \longrightarrow \mathbb{N}$ be the function defined as $f(n) = 2n$.
    - What is the image of f?
    - Is f bijective?
    - Is f invertible?