MATHEMATICAL LOGIC EXERCISES

Chiara Ghidini and **Luciano Serafini**

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We thank **Annapaola Marconi** for her work in previous editions of this booklet.

Everything should be made as simple as possible,
but not simpler.

Reader's Digest. Oct. 1977

Albert Einstein

Contents

> Mathematics is the only instructional material that can be presented in an entirely undogmatic way.
>
> *The Mathematical Intelligencer, v. 5, no. 2, 1983*
> MAX DEHN

Chapter 1

Introduction

The purpose of this booklet is to give you a number of exercises on propositional, first order and modal logics to complement the topics and exercises covered during the lectures of the course on *mathematical logic*. The material presented here is not a direct component of the course but is offered to you as an incentive and a support to understand and master the concepts and exercises presented during the course.

Symbol	Difficulty
☼	Trivial
⛅	Easy
☁	Medium
🌧	Difficult
⚡	Very difficult

When you have eliminated the impossible, what ever remains, however improbable must be the truth.

The Sign of Four.
SIR ARTHUR CONAN DOYLE

Chapter 2

Propositional Logic

2.1 Basic Concepts

Exercise 2.1.
Which of the following are well formed propositional formulas?

1. $\vee pq$
2. $(\neg(p \to (q \wedge p)))$
3. $(\neg(p \to (q = p)))$
4. $(\neg(\Diamond(q \vee p)))$
5. $(p \wedge \neg q) \vee (q \to r)$
6. $p\neg r$

Solution.

Well formed formulas: 2. and 5.

*

Exercise 2.2. ☼ ✍

*Let's consider the interpretation v where $v(p) = $ **F**, $v(q) = $ **T**, $v(r) = $ **T**. Does v satisfy the following propositional formulas?*

1. $(p \to \neg q) \lor \neg(r \land q)$
2. $(\neg p \lor \neg q) \to (p \lor \neg r)$
3. $\neg(\neg p \to \neg q) \land r$
4. $\neg(\neg p \to q \land \neg r)$

Solution.

v satisfies 1., 3. and 4.
v doesn't satisfy 2.

2.2 Truth Tables

Exercise 2.3. ☼ ✍

Compute the truth table of $(F \lor G) \land \neg(F \land G)$.

Solution.

F	G	$F \lor G$	$F \land G$	$\neg(F \land G)$	$(F \lor G) \land \neg(F \land G)$
T	*T*	*T*	*T*	*F*	**F**
T	*F*	*T*	*F*	*T*	**T**
F	*T*	*T*	*F*	*T*	**T**
F	*F*	*F*	*F*	*T*	**F**

☞ *The formula models an exclusive or!*

*

Exercise 2.4.

Use the truth tables method to determine whether $(p \to q) \vee (p \to \neg q)$ *is valid.*

Solution.

p	q	$p \to q$	$\neg q$	$p \to \neg q$	$(p \to q) \vee (p \to \neg q)$
T	*T*	*T*	*F*	*F*	**T**
T	*F*	*F*	*T*	*T*	**T**
F	*T*	*T*	*F*	*T*	**T**
F	*F*	*T*	*T*	*T*	**T**

The formula is valid since it is satisfied by every interpretation.

*

Exercise 2.5.

Use the truth tables method to determine whether $(\neg p \vee q) \wedge (q \to \neg r \wedge \neg p) \wedge (p \vee r)$ *(denoted with* φ*) is satisfiable.*

Solution.

p	q	r	$\neg p \vee q$	$\neg r \wedge \neg p$	$q \to \neg r \wedge \neg p$	$(p \vee r)$	φ
T	*T*	*T*	*T*	*F*	*F*	*T*	*F*
T	*T*	*F*	*T*	*F*	*F*	*T*	*F*
T	*F*	*T*	*F*	*F*	*T*	*T*	*F*
T	*F*	*F*	*F*	*F*	*T*	*T*	*F*
F	*T*	*T*	*T*	*F*	*F*	*T*	*F*
F	*T*	*F*	*T*	*T*	*T*	*F*	*F*
F	*F*	*T*	*T*	*F*	*T*	*T*	**T**
F	*F*	*F*	*T*	*T*	*T*	*F*	*F*

There exists an interpretation satisfying φ*, thus* φ *is satisfiable.*

*

Exercise 2.6. ☼ ✍

Use the truth tables method to determine whether the formula $\varphi : p \wedge \neg q \to p \wedge q$ *is a logical consequence of the formula* $\psi : \neg p$.

Solution.

p	q	$\neg p$	$p \wedge \neg q$	$p \wedge q$	$p \wedge \neg q \to p \wedge q$
T	*T*	*F*	*F*	*T*	*T*
T	*F*	*F*	*T*	*F*	*F*
F	*T*	**T**	*F*	*F*	**T**
F	*F*	**T**	*F*	*F*	**T**

$\psi \models \varphi$ *since each interpretation satisfying* psi *satisfies also* φ.

✻

Exercise 2.7. ☼ ✍

Use the truth tables method to determine whether $p \to (q \wedge \neg q)$ *and* $\neg p$ *are logically equivalent.*

Solution.

p	q	$q \wedge \neg q$	$p \to (q \wedge \neg q)$	$\neg p$
T	*T*	*F*	**F**	**F**
T	*F*	*F*	**F**	**F**
F	*T*	*F*	**T**	**T**
F	*F*	*F*	**T**	**T**

The two formulas are equivalent since for every possible interpretation they evaluate to tha same truth value.

✻

Exercise 2.8. ☼

Compute the truth tables for the following propositional formulas:

- $(p \to p) \to p$
- $p \to (p \to p)$
- $p \lor q \to p \land q$
- $p \lor (q \land r) \to (p \land r) \lor q$
- $p \to (q \to p)$
- $(p \land \neg q) \lor \neg(p \leftrightarrow q)$

*

Exercise 2.9.

Use the truth table method to verify whether the following formulas are valid, satisfiable or unsatisfiable:

- $(p \to q) \land \neg q \to \neg p$
- $(p \to q) \to (p \to \neg q)$
- $(p \lor q \to r) \lor p \lor q$
- $(p \lor q) \land (p \to r \land q) \land (q \to \neg r \land p)$
- $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$
- $(p \lor q) \land (\neg q \land \neg p)$
- $(\neg p \to q) \lor ((p \land \neg r) \leftrightarrow q)$
- $(p \to q) \land (p \to \neg q)$
- $(p \to (q \lor r)) \lor (r \to \neg p)$

*

Exercise 2.10.

Use the truth table method to verify whether the following logical consequences and equivalences are correct:

- $(p \to q) \models \neg p \to \neg q$
- $(p \to q) \wedge \neg q \models \neg p$
- $p \to q \wedge r \models (p \to q) \to r$
- $p \vee (\neg q \wedge r) \models q \vee \neg r \to p$
- $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- $(p \vee q) \wedge (\neg p \to \neg q) \equiv q$
- $(p \wedge q) \vee r \equiv (p \to \neg q) \to r$
- $(p \vee q) \wedge (\neg p \to \neg q) \equiv p$
- $((p \to q) \to q) \to q \equiv p \to q$

2.3 Propositional Formalization

2.3.1 Formalizing Simple Sentences

Exercise 2.11.

Let's consider a propositional language where

- p *means* "Paola is happy",
- q *means* "Paola paints a picture",
- r *means* "Renzo is happy".

Formalize the following sentences:

1. “if Paola is happy and paints a picture then Renzo isn’t happy”

2. “if Paola is happy, then she paints a picture”

3. “Paola is happy only if she paints a picture”

Solution.

1. $p \wedge q \rightarrow \neg r$

2. $p \rightarrow q$

3. $\neg(p \wedge \neg q)$ *..which is equivalent to* $p \rightarrow q$

☞ *The precision of formal languages avoid the ambiguities of natural languages.*

✻

Exercise 2.12. ☼ ✍

Let’s consider a propositional language where

- p *means* “x is a prime number”,
- q *means* “x is odd”.

Formalize the following sentences:

1. “x being prime is a sufficient condition for x being odd”

2. “x being odd is a necessary condition for x being prime”

Solution. *1. and 2.* $p \rightarrow q$

✻

Exercise 2.13. ☼ ✍

Let A =“Aldo is Italian” *and* B =“Bob is English”.
Formalize the following sentences:

1. “Aldo isn’t Italian”
2. “Aldo is Italian while Bob is English”
3. “If Aldo is Italian then Bob is not English”
4. “Aldo is Italian or if Aldo isn’t Italian then Bob is English”
5. “Either Aldo is Italian and Bob is English, or neither Aldo is Italian nor Bob is English”

Solution.

1. $\neg A$
2. $A \wedge B$
3. $A \to \neg B$
4. $A \vee (\neg A \to B)$ *logically equivalent to* $A \vee B$
5. $(A \wedge B) \vee (\neg A \wedge \neg B)$ *logically equivalent to* $A \leftrightarrow B$

✻

Exercise 2.14.

Angelo, Bruno and Carlo are three students that took the Logic exam. Let’s consider a propositional language where

- A =“Aldo passed the exam”,
- B =“Bruno passed the exam”,
- C =“Carlo passed the exam”.

Formalize the following sentences:

1. "Carlo is the only one passing the exam"
2. "Aldo is the only one not passing the exam"
3. "Only one, among Aldo, Bruno and Carlo, passed the exam"
4. "At least one among Aldo, Bruno and Carlo passed"
5. "At least two among Aldo, Bruno and Carlo passed the exam"
6. "At most two among Aldo, Bruno and Carlo passed the exam"
7. "Exactly two, among Aldo, Bruno and Carlo passed the exam"

*

Exercise 2.15. ☼ ✍

Let's consider a propositional langiage where

- A ="Angelo comes to the party",
- B ="Bruno comes to the party",
- C ="Carlo comes to the party",
- D ="Davide comes to the party".

Formalize the following sentences:

1. "If Davide comes to the party then Bruno and Carlo come too"
2. "Carlo comes to the party only if Angelo and Bruno do not come"
3. "Davide comes to the party if and only if Carlo comes and Angelo doesn't come"
4. "If Davide comes to the party, then, if Carlo doesn't come then Angelo comes"
5. "Carlo comes to the party provided that Davide doesn't come, but, if Davide comes, then Bruno doesn't come"

6. "A necessary condition for Angelo coming to the party, is that, if Bruno and Carlo aren't coming, Davide comes"

7. "Angelo, Bruno and Carlo come to the party if and only if Davide doesn't come, but, if neither Angelo nor Bruno come, then Davide comes only if Carlo comes"

Solution.

1. $D \to B \wedge C$
2. $C \to \neg A \wedge \neg B$
3. $D \leftrightarrow (C \wedge \neg A)$
4. $D \to (\neg C \to A)$
5. $(\neg D \to C) \wedge (D \to \neg B)$
6. $A \to (\neg B \wedge \neg C \to D)$
7. $(A \wedge B \wedge C \leftrightarrow \neg D) \wedge (\neg A \wedge \neg B \to (D \leftrightarrow C))$

*

Exercise 2.16.

Let's consider a propositional langiage where

- $A =$"Angelo comes to the party",
- $B =$"Bruno comes to the party",
- $C =$"Carlo comes to the party",
- $D =$"Davide comes to the party".

Formalize the following sentences:

1. "Angelo comes to the party while Bruno doesn't"

2. "Either Carlo comes to the party, or Bruno and Davide don't come"

3. "If Angelo and Bruno come to the party, then Carlo comes provided that Davide doesn't come"

4. "Carlo comes to the party if Bruno and Angelo don't come, or if Davide comes"

5. "If Angelo comes to the party then Bruno or Carlo come too, but if Angelo doesn't come to the party, then Carlo and Davide come"

✻

Exercise 2.17. ☼ ✍

Socrate says:

> "If I'm guilty, I must be punished;
> I'm guilty. Thus I must be punished."

Is the argument logically correct?

Solution. *The argument is logically correct: if* p *means* "I'm guilty" *and* q *means* "I must be punished"*, then:*

$(p \to q) \land p \models q$ *(modus ponens)*

✻

Exercise 2.18. ☼ ✍

Socrate says:

> "If I'm guilty, I must be punished;
> I'm not guilty. Thus I must not be punished."

Is the argument logically correct?

Solution. *The argument is not logically correct:*

$(p \to q) \wedge \neg p \not\models \neg q$

☞ *consider for instance* $v(p) = \boldsymbol{F}$ *and* $v(q) = \boldsymbol{T}$

*

Exercise 2.19. ☼

Socrate says:

"If I'm guilty, I must be punished;
I must not be punished. Thus I'm not guilty."

Is the argument logically correct?

*

Exercise 2.20. ☼

Socrate says:

"If I'm guilty, I must be punished;
I must be punished. Thus I'm guilty."

Is the argument logically correct?

*

Exercise 2.21. ☼

Formalize the following arguments and verify whether they are correct:

- "If Carlo won the competition, then either Mario came second or Sergio came third. Sergio didn't come third. Thus, if Mario didn't come second, then Carlo didn't win the competition."

- “If Carlo won the competition, then either Mario came second or Sergio came third. Mario didn’t come second. Thus, if Carlo won the competition, then Sergio didn’t come third.”
- “If Carlo won the competition, then Mario came second and Sergio came third. Mario didn’t come second. Thus Carlo didn’t win the competition.”
- “If Carlo won the competition, then, if Mario came second then Sergio came third. Mario didn’t come second. Thus, either Carlo won or Sergio arrived third”
- “If you play and you study you’ll pass the exams, while if you play and don’t study you won’t pass. Thus, if you play, either you study and you’ll pass the exams, or you don’t study and you won’t pass.”

2.3.2 Formalizing Problems

Exercise 2.22.
Aladdin finds two trunks A and B in a cave. He knows that each of them either contains a treasure or a fatal trap.

> *On trunk A is written:* “At least one of these two trunks contains a treasure.”
>
> *On trunk B is written:* “In A there’s a fatal trap.”

Aladdin knows that either both the inscriptions are true, or they are both false. Can Aladdin choose a trunk being sure that he will find a treasure?
If this is the case, which trunk should he open?

Solution. *Let’s consider a propositional language where $a =$*“Trunk A contains the treasure” *and $b =$*“Trunk B contains the treasure”.

☞ *Obviously* $\neg a$ =“Trunk a contains a trap” *(and similarly for* $\neg b$*), since each trunk either contains a treasure or a trap (exclusive or).*

Let's formalize what Aladdin knows:

- *Formalization of the inscriptions:*

  $a \vee b$ “At least one of these two trunks contains a treasure.”

  $\neg a$ “A contains a trap”

- *Formalization of the problem:*

  1. “either both the inscriptions are true, or they are both false”

  $(a \vee b) \leftrightarrow \neg a$

What we can do is to verify whether there is any interpretation satisfying the formula in 1. :

- *The only interpretation satisfying 1. is:*

  $v(a) = \boldsymbol{F}$ *and* $v(b) = \boldsymbol{T}$

- *Thus Aladdin can open trunk* B*, being sure that it contains a treasure.*

*

Exercise 2.23.

Suppose we know that:

- “if Paolo is thin, then Carlo is not blonde or Roberta is not tall”
- “if Roberta is tall then Sandra is lovely”
- “if Sandra is lovely and Carlo is blonde then Paolo is thin”
- “Carlo is blonde”

Can we deduce that “Roberta is not tall” *?*

*

Exercise 2.24.

Three boxes are presented to you. One contains gold, the other two are empty. Each box has imprinted on it a clue as to its contents; the clues are:

Box 1 "The gold is not here"

Box 2 "The gold is not here"

Box 3 "The gold is in Box 2"

Only one message is true; the other two are false. Which box has the gold?

Formalize the puzzle in Propositional Logic and find the solution using a truth table.

Solution. *Let* B_i *with* $i \in \{1, 2, 3\}$ *stand for "gold is in the* i*-th box". We can formalize the statements of the problem as follows:*

1. *One box contains gold, the other two are empty.*

$$(B_1 \wedge \neg B_2 \wedge \neg B_3) \vee (\neg B_1 \wedge B_2 \wedge \neg B_3) \vee (\neg B_1 \wedge \neg B_2 \wedge B_3) \qquad (2.1)$$

2. *Only one message is true; the other two are false.*

$$(\neg B_1 \wedge \neg\neg B_2 \wedge \neg B_2) \vee (\neg\neg B_1 \wedge \neg B_2 \wedge \neg B_2) \vee (\neg\neg B_1 \wedge \neg\neg B_2 \wedge B_2) \qquad (2.2)$$

(2.2) is equivalent to:

$$(B_1 \wedge \neg B_2) \vee (B_1 \wedge B_2) \qquad (2.3)$$

Let us compute the truth table for (2.1) and (2.3)

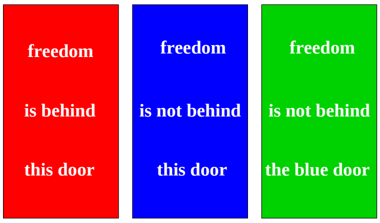
B_1	B_2	B_3	(2.1)	(2.3)
T	T	T	F	T
T	T	F	F	T
T	F	T	F	T
T	**F**	**F**	**T**	**T**
F	T	T	F	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

The only assignment I that verifies both (2.1) and (2.3) is the one with $I(B_1) = T$ and $I(B_2) = I(B_3) = F$, which implies that the gold is in the first box.

*

Exercise 2.25.

Kyle, Neal, and Grant find themselves trapped in a dark and cold dungeon (HOW they arrived there is another story). After a quick search the boys find three doors, the first one red, the second one blue, and the third one green.
Behind one of the doors is a path to freedom. Behind the other two doors, however, is an evil fire-breathing dragon. Opening a door to the dragon means almost certain death.
On each door there is an inscription:



Given the fact that at LEAST ONE of the three statements on the three doors is true and at LEAST ONE of them is false, which door would lead the boys to safety?

Solution.

Language

- r: "freedom is behind the red door"
- b: "freedom is behind the blue door"
- g: "freedom is behind the green door"

Axioms

1. “behind one of the door is a path to freedom, behind the other two doors is an evil dragon”

$$(r \wedge \neg b \wedge \neg g) \vee (\neg r \wedge b \wedge \neg g) \vee (\neg r \wedge \neg b \wedge g) \tag{2.4}$$

2. “at least one of the three statements is true”

$$r \vee \neg b \tag{2.5}$$

3. “at least one of the three statements is false”

$$\neg r \vee b \tag{2.6}$$

Solution

r	b	g	*2.5*	*2.6*	*2.5* $\wedge$ *2.6*
T	*F*	*F*	*T*	*F*	*F*
F	*T*	*F*	*F*	*T*	*F*
F	*F*	*T*	*T*	*T*	*T*

Freedom is behind the **green door!**

*

Exercise 2.26. ✍

The Labyrinth Guardians.
You are walking in a labyrinth and all of a sudden you find yourself in front of three possible roads: the road on your left is paved with gold, the one in front of you is paved with marble, while the one on your right is made of small stones. Each street is protected by a guardian. You talk to the guardians and this is what they tell you:

- *The guardian of the gold street:* “This road will bring you straight to the center. Moreover, if the stones take you to the center, then also the marble takes you to the center.”
- *The guardian of the marble street:* “Neither the gold nor the stones will take you to the center.”
- *The guardian of the stone street:* “Follow the gold and you’ll reach the center, follow the marble and you will be lost.”

Given that you know that all the guardians are liars, can you choose a road being sure that it will lead you to the center of the labyrinth? If this is the case, which road you choose?

Provide a propositional language and a set of axioms that formalize the problem and show whether you can choose a road being sure it will lead to the center.

Solution.

Language

- g*:* “the gold road brings to the center”
- m*:* “the marble road brings to the center”
- s*:* “the stone road brings to the center”

Axioms

1. “The guardian of the gold street is a liar”

$$\neg(g \wedge (s \rightarrow m)) \tag{2.7}$$

which can be simplified to obtain

$$\neg g \vee (s \wedge \neg m)$$

2. “The guardian of the marble street is a liar”

$$\neg(\neg g \wedge \neg s) \tag{2.8}$$

which can be simplified to obtain

$$g \vee s$$

3. "The guardian of the stone street is a liar"

$$\neg(g \wedge \neg m) \tag{2.9}$$

which can be simplified to obtain

$$\neg g \vee m$$

Solution

g	m	s	2.7	2.8	2.9	2.7 ∧ 2.8 ∧ 2.9
1	1	1	0	1	1	0
1	1	0	0	1	1	0
1	0	1	1	1	0	0
1	0	0	0	1	0	0
0	1	**1**	1	1	1	**1**
0	1	0	1	0	1	0
0	0	**1**	1	1	1	**1**
0	0	0	1	0	1	0

We have two possible interpretations that satisfy the axioms,
and in both of them the stone street brings to the center.
Thus I can choose the **stone street** being sure that it leads to the center.

✻

Exercise 2.27.

Consider the finite set of binary strings

$$\left\{ \begin{array}{l} (000000), (100000), (110000), (111000), (111100), (111110), \\ (111111), (011111), (001111), (000111), (000011), (000001) \end{array} \right\}$$

Explain how it is possible to represent such a set in a propositional formula and find the most compact representation.

Solution.

Language *For each* $0 \leq i \leq 5$*,* b_i *is a proposition, which intuitively means that the* i*-th bit has value* 1*. Obviously,* $\neg b_i$ *means that the* i*-th bit does not have value* 1*, and thus it has value* 0*.*

Axioms *A possible (compact) representation of the finite set of binary strings is given by the following formula:*

$$\bigvee_{k=0}^{5}\left(\left(\bigwedge_{i=0}^{k}\neg b_i \wedge \bigwedge_{i=k+1}^{5} b_i\right) \vee \left(\bigwedge_{i=0}^{k} b_i \wedge \bigwedge_{i=k+1}^{5}\neg b_i\right)\right) \tag{2.10}$$

✻

Exercise 2.28.

Provide a propositional language and a set of axioms that formalize the **graph coloring problem** *of a graph with at most* n *nodes, with connection degree* $\leq m$*, and with less then* $k+1$ *colors.*

- node degree*: number of adjacent nodes*
- connection degree *of a graph: max among all the degree of its nodes*
- graph coloring problem*: given a non-oriented graph, associate a color to each of its nodes in such a way that no pair of adjacent nodes have the same color.*

Solution.

Language

- *For each* $1 \leq i \leq n$ *and* $1 \leq c \leq k$*,* color_{ic} *is a proposition, which intuitively means that* "the i-th node has the c color"
- *For each* $1 \leq i \neq j \leq n$*,* edge_{ij} *is a proposition, which intuitively means that* "the i-th node is connected with the j-th node".

Axioms

1. *for each* $1 \leq i \leq n$, $\bigvee_{c=1}^{k} \mathsf{color}_{ic}$
   "each node has at least one color"

2. *for each* $1 \leq i \leq n$ *and* $1 \leq c, c' \leq k$, $\mathsf{color}_{ic} \rightarrow \neg\mathsf{color}_{ic'}$
   "every node has at most 1 color"

3. *for each* $1 \leq i, j \leq n$ *and* $1 \leq c \leq k$, $\mathsf{edge}_{ij} \rightarrow \neg(\mathsf{color}_{ic} \wedge \mathsf{color}_{jc})$
   "adjacent nodes do not have the same color"

4. *for each* $1 \leq i \leq n$, *and each* $J \subseteq \{1..n\}$, *where* $|J| = m$, $\bigwedge_{j \in J} \mathsf{edge}_{ij} \rightarrow \bigwedge_{j \notin J} \neg\mathsf{edge}_{ij}$
   "every node has at most m connected node"

*

Exercise 2.29.
Anna and Barbara carpool to work. On any day, either Anna drives Barbara or Barbara drives Anna. In the former case, Anna is the driver and Barbara is the passenger; in the latter case Barbara is the driver and Anna is the passenger.
Formalize the problem using the following propositions:

1. Anna drives Barbara
2. Barbara drives Anna
3. Anna is the driver
4. Barbara is the driver
5. Anna is the passenger
6. Barbara is the passenger

*

Exercise 2.30.

Define a propositional language which allows to describe the state of a traffic light on different instants.
With the language defined above provide a (set of) formulas which expresses the following facts:

1. *the traffic light is either green, or red or orange;*
2. *the traffic light switches from green to orange, from orange to red, and from red to green;*
3. *it can keep the same color over at most 3 successive states.*

Solution.

Language

- g_k="traffic light is green at instant k"
- r_k="traffic light is red at instant k"
- o_k="traffic light is orange at instant k"

Axioms

1. "the traffic light is either green, or red or orange"
   $(g_k \leftrightarrow (\neg r_k \wedge \neg o_k)) \wedge (r_k \leftrightarrow (\neg g_k \wedge \neg o_k)) \wedge (o_k \leftrightarrow (\neg r_k \wedge \neg g_k))$
2. "the traffic light switches from green to orange, from orange to red, and from red to green"
   $(g_{k-1} \rightarrow (g_k \vee o_k)) \wedge (o_{k-1} \rightarrow (o_k \vee r_k)) \wedge (r_{k-1} \rightarrow (r_k \vee g_k))$
3. "it can keep the same color over at most 3 successive states"
   $(g_{k-3} \wedge g_{k-2} \wedge g_{k-1} \rightarrow \neg g_k) \wedge (r_{k-3} \wedge r_{k-2} \wedge r_{k-1} \rightarrow \neg r_k) \wedge (o_{k-3} \wedge o_{k-2} \wedge o_{k-1} \rightarrow \neg o_k)$

*

Exercise 2.31.

Sudoku is a placement puzzle. The aim of the puzzle is to enter a numeral from 1 through 9 in each cell of a grid, most frequently a 9×9 grid made up of 3×3 subgrids (called "regions"), starting with various numerals given in some cells (the "givens"). Each row, column and region must contain only one instance of each numeral. Its grid layout is like the one shown in the following schema

		9				7		
	4		5		9		1	
3				1				2
	1			6			7	
		2	7		1	8		
	5			4			3	
7				3				4
	8		2		4		6	
		6				5		

Provide a formalization in propositional logic of the Sudoku problem, so that any truth assignment to the propositional variables that satisfy the axioms is a solution for the puzzle.

Solution.

Language *For $1 \leq n, r, c \leq 9$, we define the proposition*

$$\mathit{in}(n, r, c)$$

which means that the number n has been inserted in the cross between row r and column c.

Axioms

1. "A raw contains all numbers from 1 to 9 "

$$\bigwedge_{r=1}^{9} \left(\bigwedge_{n=1}^{9} \left(\bigvee_{c=1}^{9} \mathit{in}(n, r, c) \right) \right)$$

2. "A column contains all numbers from 1 to 9"

$$\bigwedge_{c=1}^{9}\left(\bigwedge_{n=1}^{9}\left(\bigvee_{r=1}^{9} \mathit{in}(n,r,c)\right)\right)$$

3. "A region (sub-grid) contains all numbers from 1 to 9"

$$\textit{for any} \quad 0 \leq k, h \leq 2 \qquad \bigwedge_{n=1}^{9}\left(\bigvee_{r=1}^{3}\left(\bigvee_{c=1}^{3} \mathit{in}(n, 3*k+r, 3*h+c)\right)\right)$$

4. "A cell cannot contain two numbers"

$$\textit{for any} \;\; 1 \leq n, n', c, r \leq 9 \;\textit{and}\; n \neq n' \qquad \mathit{in}(n,r,c) \rightarrow \neg \mathit{in}(n',r,c)$$

2.4 Normal Form Reduction

Exercise 2.32. ☼ ✍

Reduce to Negative Normal Form (NNF) the formula

$$\neg(\neg p \vee q) \vee (r \rightarrow \neg s)$$

Solution.

1. $\neg(\neg p \vee q) \vee (\neg r \vee \neg s)$

2. $(\neg\neg p \wedge \neg q) \vee (\neg r \vee \neg s)$

3. $(p \wedge \neg q) \vee (\neg r \vee \neg s)$

*

Exercise 2.33. ☼ ✍

Reduce to NNF the formula

$$(\neg p \to q) \to (q \to \neg r)$$

Solution.

1. $\neg(\neg p \to q) \vee (q \to \neg r)$
2. $\neg(p \vee q) \vee (\neg q \vee \neg r)$
3. $(\neg p \wedge \neg q) \vee (\neg q \vee \neg r)$

*

Exercise 2.34. ☼ ✍

Reduce to Conjunctive Normal Form (CNF) the formula

$$\neg(\neg p \vee q) \vee (r \to \neg s)$$

Solution.

1. $\neg(\neg p \vee q) \vee (\neg r \vee \neg s)$
2. $(\neg\neg p \wedge \neg q) \vee (\neg r \vee \neg s)$
3. $(p \wedge \neg q) \vee (\neg r \vee \neg s)$ NNF
4. $(p \vee \neg r \vee \neg s) \wedge (\neg q \vee \neg r \vee \neg s)$

*

Exercise 2.35. ☼ ✍

Reduce to CNF the formula

$$(\neg p \to q) \to (q \to \neg r)$$

Solution.

1. $\neg(\neg p \to q) \vee (q \to \neg r)$

2. $\neg(p \vee q) \vee (\neg q \vee \neg r)$

3. $(\neg p \wedge \neg q) \vee (\neg q \vee \neg r)$ **NNF**

4. $(\neg p \vee \neg q \vee \neg r) \wedge (\neg q \vee \neg r)$

*

Exercise 2.36.

Reduce to CNF the following formulas:

- $p \to (q \wedge r)$
- $(p \vee q) \to r$
- $\neg(\neg p \vee q) \vee (r \to \neg s)$
- $\neg((p \to (q \to r))) \to ((p \to q) \to (p \to r))$
- $p \vee (\neg q \wedge (r \to \neg p))$
- $\neg((((a \to b)) \to a) \to a)$
- $\neg(a \vee (a \to b))$

Mathematics as a science, commenced when first someone, probably a Greek, proved propositions about "any" things or about "some" things, without specifications of definite particular things.

ALFRED NORTH WHITEHEAD

Chapter 3

First Order Logic

3.1 Basic Concepts

Exercise 3.1.
Non Logical symbols:

constants a, b*; functions* f^1, g^2*; predicates* p^1, r^2, q^3.

Say whether the following strings of symbols are well formed FOL formulas or terms:

1. $q(a)$
2. $p(y)$
3. $p(g(b))$
4. $\neg r(x, a)$
5. $q(x, p(a), b)$
6. $p(g(f(a), g(x, f(x))))$
7. $q(f(a), f(f(x)), f(g(f(z), g(a, b))))$
8. $r(a, r(a, a))$

Solution.

Well formed formulas: 2., 4., 6., and 7.
All other strings are NOT well formed FOL formulas nor terms.

*

Exercise 3.2.
Non Logical symbols:

constants a, b*; functions* f^1, g^2*; predicates* p^1, r^2, q^3.

Say whether the following strings of symbols are well formed FOL formulas or terms:

1. $r(a, g(a, a))$;
2. $g(a, g(a, a))$;
3. $\forall x.\neg p(x)$;
4. $\neg r(p(a), x)$;
5. $\exists a.r(a, a)$;
6. $\exists x.q(x, f(x), b) \rightarrow \forall x.r(a, x)$;
7. $\exists x.p(r(a, x))$;
8. $\forall r(x, a)$;

Solution.

Well formed formulas: 1., 3., and 6.
Well formed terms: 2.
All other strings are NOT well formed FOL formulas nor terms.

*

Exercise 3.3. ☼ ✍

Non Logical symbols:

constants a, b; *functions* f^1, g^2; *predicates* p^1, r^2, q^3.

Say whether the following strings of symbols are well formed FOL formulas or terms:

1. $a \rightarrow p(b)$;
2. $r(x, b) \rightarrow \exists y.q(y, y, y)$;
3. $r(x, b) \vee \neg\exists y.g(y, b)$;
4. $\neg y \vee p(y)$;
5. $\neg\neg p(a)$;
6. $\neg\forall x.\neg p(x)$;
7. $\forall x\exists y.(r(x, y) \rightarrow r(y, x))$;
8. $\forall x\exists y.(r(x, y) \rightarrow (r(y, x) \vee (f(a) = g(a, x))))$;

Solution.

Well formed formulas: 2., 4., 5., 6., 7., and 8.
All other strings are NOT well formed FOL formulas nor terms.

*

Exercise 3.4. ☼ ✍

Find free variables in the following formulas:

1. $p(x) \wedge \neg r(y, a)$
2. $\exists x.r(x, y)$
3. $\forall x.p(x) \rightarrow \exists y.\neg q(f(x), y, f(y))$
4. $\forall x\exists y.r(x, f(y))$

5. $\forall x \exists y. r(x, f(y)) \to r(x, y)$

Solution.

1. *x,y free*
2. *y free*
3. *x free*
4. *no free variables*
5. *x,y free*

*

Exercise 3.5. ☼ ✍

Find free variables in the following formulas:

1. $\forall x.(p(x) \to \exists y.\neg q(f(x), y, f(y)))$
2. $\forall x(\exists y.r(x, f(y)) \to r(x, y))$
3. $\forall z.(p(z) \to \exists y.(\exists x.q(x, y, z) \vee q(z, y, x)))$
4. $\forall z \exists u \exists y.(q(z, u, g(u, y)) \vee r(u, g(z, u)))$
5. $\forall z \exists x \exists y(q(z, u, g(u, y)) \vee r(u, g(z, u)))$

Solution.

1. *no free variables*
2. *y free*
3. *x free*
4. *no free variables*
5. *u free*

3.2 FOL Formalization

Exercise 3.6.

What is the meaning of the followinf FOL formulas?

1. $bought(Frank, dvd)$
2. $\exists x.bought(Frank, x)$
3. $\forall x.(bought(Frank, x) \rightarrow bought(Susan, x))$
4. $\forall x.bought(Frank, x) \rightarrow \forall x.bought(Susan, x)$
5. $\forall x \exists y.bought(x, y)$
6. $\exists x \forall y.bought(x, y)$

Solution.

1. *"Frank bought a dvd."*
2. *"Frank bought something."*
3. *"Susan bought everything that Frank bought."*
4. *"If Frank bought everything, so did Susan."*
5. *"Everyone bought something."*
6. *"Someone bought everything."*

*

Exercise 3.7.

Which of the following formulas is a formalization of the sentence:

"There is a computer which is not used by any student"

- $\exists x.(Computer(x) \wedge \forall y.(\neg Student(y) \wedge \neg Uses(y,x)))$
- $\exists x.(Computer(x) \rightarrow \forall y.(Student(y) \rightarrow \neg Uses(y,x)))$
- $\exists x.(Computer(x) \wedge \forall y.(Student(y) \rightarrow \neg Uses(y,x)))$

*

Exercise 3.8.

Define an appropriate language and formalize the following sentences using FOL formulas.

1. *All Students are smart.*
2. *There exists a student.*
3. *There exists a smart student.*
4. *Every student loves some student.*
5. *Every student loves some other student.*
6. *There is a student who is loved by every other student.*
7. *Bill is a student.*
8. *Bill takes either Analysis or Geometry (but not both).*
9. *Bill takes Analysis and Geometry.*
10. *Bill doesn't take Analysis.*
11. *No students love Bill.*

Solution.

1. $\forall x.(Student(x) \rightarrow Smart(x))$
2. $\exists x.Student(x)$
3. $\exists x.(Student(x) \wedge Smart(x))$

4. $\forall x.(Student(x) \rightarrow \exists y.(Student(y) \wedge Loves(x, y)))$

5. $\forall x.(Student(x) \rightarrow \exists y.(Student(y) \wedge \neg(x = y) \wedge Loves(x, y)))$

6. $\exists x.(Student(x) \wedge \forall y.(Student(y) \wedge \neg(x = y) \rightarrow Loves(y, x)))$

7. $Student(Bill)$

8. $Takes(Bill, Analysis) \leftrightarrow \neg Takes(Bill, Geometry)$

9. $Takes(Bill, Analysis) \wedge Takes(Bill, Geometry)$

10. $\neg Takes(Bill, Analysis)$

11. $\neg \exists x.(Student(x) \wedge Loves(x, Bill))$

*

Exercise 3.9.

Define an appropriate language and formalize the following sentences using FOL formulas.

1. *Bill has at least one sister.*
2. *Bill has no sister.*
3. *Bill has at most one sister.*
4. *Bill has (exactly) one sister.*
5. *Bill has at least two sisters.*
6. *Every student takes at least one course.*
7. *Only one student failed Geometry.*
8. *No student failed Geometry but at least one student failed Analysis.*
9. *Every student who takes Analysis also takes Geometry.*

Solution.

1. $\exists x.SisterOf(x, Bill)$

2. $\neg\exists x.SisterOf(x, Bill)$

3. $\forall x\forall y.(SisterOf(x, Bill) \wedge SisterOf(y, Bill) \rightarrow x = y)$

4. $\exists x.(SisterOf(x, Bill) \wedge \forall y.(SisterOf(y, Bill) \rightarrow x = y))$

5. $\exists x\exists y.(SisterOf(x, Bill) \wedge SisterOf(y, Bill) \wedge \neg(x = y))$

6. $\forall x.(Student(x) \rightarrow \exists y.(Course(y) \wedge Takes(x, y)))$

7. $\exists x.(Student(x) \wedge Failed(x, Geometry) \wedge \forall y.(Student(y) \wedge Failed(y, Geometry) \rightarrow x = y))$

8. $\neg\exists x.(Student(x) \wedge Failed(x, Geometry)) \wedge \exists x.(Student(x) \wedge Failed(x, Analysis))$

9. $\forall x.(Student(x) \wedge Takes(x, Analysis) \rightarrow Takes(x, Geometry))$

*

Exercise 3.10.
Define an appropriate language and formalize the following sentences in FOL:

- *someone likes Mary.*
- *nobody likes Mary.*
- *nobody loves Bob but Bob loves Mary.*
- *if David loves someone, then he loves Mary.*
- *if someone loves David, then he (someone) loves also Mary.*
- *everybody loves David or Mary.*

*

Exercise 3.11.

Define an appropriate language and formalize the following sentences in FOL:

- *there is at least one person who loves Mary.*
- *there is at most one person who loves Mary.*
- *there is exactly one person who loves Mary.*
- *there are exactly two persons who love Mary.*
- *if Bob loves everyone that Mary loves, and Bob loves David, then Mary doesn't love David.*
- *Only Mary loves Bob.*

*

Exercise 3.12.

Define an appropriate language and formalize the following sentences in FOL:

1. *"A is above C, D is on E and above F."*
2. *"A is green while C is not."*
3. *"Everything is on something."*
4. *"Everything that is free has nothing on it."*
5. *"Everything that is green is free."*
6. *"There is something that is red and is not free."*
7. *"Everything that is not green and is above B, is red."*

Solution.

Language *Constants:* A, B, C, D, E, F;
Predicates: $On^2, Above^2, Free^1, Red^1, Green^1$.

Axioms

1. *"A is above C, D is above F and on E."*
   $\phi_1 : Above(A, C) \wedge Above(E, F) \wedge On(D, E)$
2. *"A is green while C is not."*
   $\phi_2 : Green(A) \wedge \neg Green(C)$
3. *"Everything is on something."*
   $\phi_3 : \forall x \exists y. On(x, y)$
4. *"Everything that is free has nothing on it."*
   $\phi_4 : \forall x.(Free(x) \rightarrow \neg \exists y. On(y, x))$
5. *"Everything that is green is free."*
   $\phi_5 : \forall x.(Green(x) \rightarrow Free(x))$
6. *"There is something that is red and is not free."*
   $\phi_6 : \exists x.(Red(x) \wedge \neg Free(x))$
7. *"Everything that is not green and is above B, is red."*
   $\phi_7 : \forall x.(\neg Green(x) \wedge Above(x, B) \rightarrow Red(x))$

*

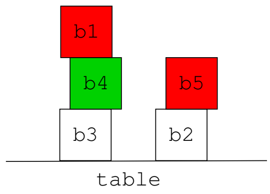
Exercise 3.13.

Language *Constants:* A, B, C, D, E, F*;*
Predicates: $On^2, Above^2, Free^1, Red^1, Green^1$.

Interpretation *Let interpretation* $\mathcal{I}_1$ *be the following:*

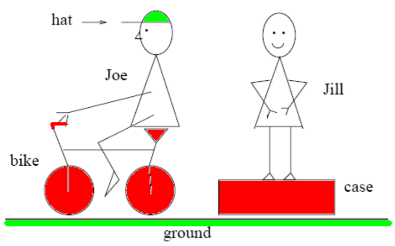
- $\mathcal{I}_1(A) = b_1$, $\mathcal{I}_1(B) = b_2$, $\mathcal{I}_1(C) = b_3$, $\mathcal{I}_1(D) = b_4$, $\mathcal{I}_1(E) = b_5$, $\mathcal{I}_1(F) = table$
- $\mathcal{I}_1(On) = \{\langle b_1, b_4\rangle, \langle b_4, b_3\rangle, \langle b_3, table\rangle, \langle b_5, b_2\rangle, \langle b_2, table\rangle\}$
- $\mathcal{I}_1(Above) = \{\langle b_1, b_4\rangle, \langle b_1, b_3\rangle, \langle b_1, table\rangle, \langle b_4, b_3\rangle, \langle b_4, table\rangle, \langle b_3, table\rangle, \langle b_5, b_2\rangle, \langle b_5, table\rangle, \langle b_2, table\rangle\}$

- $\mathcal{I}_1(Free) = \{\langle b_1 \rangle, \langle b_5 \rangle\}$, $\mathcal{I}_1(Green) = \{\langle b_4 \rangle\}$, $\mathcal{I}_1(Red) = \{\langle b_1 \rangle, \langle b_5 \rangle\}$



And let interpretation $\mathcal{I}_2$ be:

- $\mathcal{I}_2(A) = hat$, $\mathcal{I}_2(B) = Joe$, $\mathcal{I}_2(C) = bike$, $\mathcal{I}_2(D) = Jill$, $\mathcal{I}_2(E) = case$, $\mathcal{I}_2(F) = ground$
- $\mathcal{I}_2(On) = \{\langle hat, Joe \rangle, \langle Joe, bike \rangle, \langle bike, ground \rangle, \langle Jill, case \rangle, \langle case, ground \rangle\}$
- $\mathcal{I}_2(Above) = \{\langle hat, Joe \rangle, \langle hat, bike \rangle, \langle hat, ground \rangle, \langle Joe, bike \rangle, \langle Joe, ground \rangle, \langle bike, ground \rangle, \langle Jill, case \rangle, \langle Jill, ground \rangle, \langle case, ground \rangle\}$
- $\mathcal{I}_2(Free) = \{\langle hat \rangle, \langle Jill \rangle\}$, $\mathcal{I}_2(Green) = \{\langle hat \rangle, \langle ground \rangle\}$, $\mathcal{I}_2(Red) = \{\langle bike \rangle, \langle case \rangle\}$



For each formula in Exercise 3.12, decide whether it is satisfied by $\mathcal{I}_1$ and/or $\mathcal{I}_2$.

Solution.

- $\mathcal{I}_1 \models \neg\phi_1 \wedge \neg\phi_2 \wedge \neg\phi_3 \wedge \phi_4 \wedge \neg\phi_5 \wedge \neg\phi_6 \wedge \phi_7$
- $\mathcal{I}_2 \models \phi_1 \wedge \phi_2 \wedge \neg\phi_3 \wedge \phi_4 \wedge \neg\phi_5 \wedge \phi_6 \wedge \phi_7$

*

Exercise 3.14. ☼ ✍

Consider the following sentences:

1. *All actors and journalists invited to the party are late.*
2. *There is at least a person who is on time.*
3. *There is at least an invited person who is neither a journalist nor an actor.*

Formalize the sentences and prove that 3. is not a logical consequence of 1. and 2.

Solution.

1. $\forall x.((a(x) \vee j(x)) \wedge i(x) \rightarrow l(x))$
2. $\exists x.\neg l(x)$
3. $\exists x.(i(x) \wedge \neg a(x) \wedge \neg j(x))$

It's sufficient to find an interpretation $\mathcal{I}$ for which the logical consequence does not hold:

	l(x)	a(x)	j(x)	i(x)
Bob	F	T	F	F
Tom	T	T	F	T
Mary	T	F	T	T

*

Exercise 3.15. ☼

Let $\Delta = \{1, 3, 5, 15\}$ and $\mathcal{I}$ be an interpretation on Δ interpreting the predicate symbols E^1 as 'being even', M^2 as 'being a multiple of' and L^2 as 'being less then', and s.t. $\mathcal{I}(a) = 1, \mathcal{I}(b) = 3, \mathcal{I}(c) = 5, \mathcal{I}(d) = 15$.
Determine whether $\mathcal{I}$ satisfies the following formulas:

1. $\exists y.E(y)$
2. $\forall x.\neg E(x)$
3. $\forall x.M(x,a)$
4. $\forall x.M(x,b)$
5. $\exists x.M(x,d)$
6. $\exists x.L(x,a)$
7. $\forall x.(E(x) \rightarrow M(x,a))$
8. $\forall x \exists y.L(x,y)$
9. $\forall x \exists y.M(x,y)$
10. $\forall x.(M(x,b) \rightarrow L(x,c))$
11. $\forall x \forall y.(L(x,y) \rightarrow \neg L(y,x))$
12. $\forall x.(M(x,c) \vee L(x,c))$

*

Exercise 3.16.

Provide a FOL language and a set of axioms that formalize the **graph coloring problem** *of a graph with at most n nodes, with connection degree $\leq m$, and with less then $k+1$ colors.*

- node degree*: number of adjacent nodes*
- connection degree *of a graph: max among all the degree of its nodes*
- graph coloring problem*: given a non-oriented graph, associate a color to each of its nodes in such a way that no pair of adjacent nodes have the same color.*

Solution.

Language

- *A unary function* color*, where* color(x) *is the color associated to the node* x
- *A unary predicate* node*, where* node(x) *means that* x *is a node*
- *A binary predicate* edge*, where* edge(x, y) *means that* x *is connected to* y

Axioms

1. $\forall x \forall y.(\mathsf{edge}(x, y) \rightarrow (\mathsf{color}(x) \neq \mathsf{color}(y))$
   "Two connected node are not equally colored."
2. $\forall x \forall x_1 \ldots \forall x_{k+1}.\left(\bigwedge_{h=1}^{k+1} \mathsf{edge}(x, x_h) \rightarrow \bigvee_{i,j=1,j\neq i}^{k+1} x_i = x_j\right)$
   "A node does not have more than k connected nodes."

*

Exercise 3.17.

Let $\{c_1, .., c_k\}$ *be a non empty and finite set of colors. A partially colored directed graph is a structure* $\langle N, R, C\rangle$ *where*

- N *is a non empty set of nodes*
- R *is a binary relation on* N
- C *associates colors to nodes (not all the nodes are necessarily colored,. and each node has at most one color)*

Provide a first order language and a set of axioms that formalize partially colored graphs. Show that every model of this theory correspond to a partially colored graph, and vice-versa. For each of the following properties, write a formula which is true in all and only the graphs that satisfies the property:

1. *connected nodes don't have the same color*
2. *the graph contains only 2 yellow nodes*

3. *starting from a red node one can reach in at most 4 steps a green node*
4. *for each color there is at least a node with this color*
5. *the graph is composed of $|C|$ disjoint non empty subgraphs, one for each color*

Solution.

Language

- *a binary predicate* edge*, where* $\mathsf{edge}(n, m)$ *means that node* n *is connected to node* m
- *a binary predicate* color*, where* $\mathsf{color}(n, x)$ *means that node* n *has color* x
- *the following constants:* yellow, green, red

Axioms

0. "each node has at most one color"

$$\forall n \forall x.(\mathsf{color}(n, x) \rightarrow \neg\exists y.(y \neq x \wedge \mathsf{color}(n, y)))$$

1. "connected nodes do not have the same color"

$$\forall n \forall m \forall x.(\mathsf{edge}(n, m) \wedge \mathsf{color}(n, x) \rightarrow \neg\mathsf{color}(m, x))$$

2. "the graph contains only two yellow nodes"

$$\exists n \exists n'.(\mathsf{color}(n, \mathsf{yellow}) \wedge \mathsf{color}(n', \mathsf{yellow}) \wedge n \neq n' \wedge \\ \forall m.(m \neq n \wedge m \neq n' \rightarrow \neg\mathsf{color}(m, \mathsf{yellow})))$$

3. "starting from a red node one can reach in at most 4 steps a green node"

$$\forall n(\mathsf{color}(n, \mathsf{red}) \rightarrow \\ (\exists n_1.(\mathsf{edge}(n, n_1) \wedge \mathsf{color}(n_1, \mathsf{green})) \vee \\ \exists n_1, n_2.(\mathsf{edge}(n, n_1) \wedge \mathsf{edge}(n_1, n_2) \wedge \mathsf{color}(n_2, \mathsf{green})) \vee \\ \exists n_1, n_2, n_3.(\mathsf{edge}(n, n_1) \wedge \mathsf{edge}(n_1, n_2) \wedge \mathsf{edge}(n_2, n_3) \wedge \mathsf{color}(n_3, \mathsf{green})) \vee \\ \exists n_1, n_2, n_3, n_4.(\mathsf{edge}(n, n_1) \wedge \mathsf{edge}(n_1, n_2) \wedge \mathsf{edge}(n_2, n_3) \wedge \mathsf{edge}(n_3, n_4) \wedge \mathsf{color}(n_4, \mathsf{green})) \\ ))$$

4. "for each color there is at least a node with this color"

$$\forall x \exists n.\mathsf{color}(n, x)$$

5. "the graph is composed of $|C|$ disjoint non empty subgraphs, one for each color"

$$\begin{aligned}
&\forall x \exists n.\mathsf{color}(n, x) \land \\
&\forall n \exists x.\mathsf{color}(n, x) \land \\
&\forall n \forall x.(\mathsf{color}(n, x) \rightarrow \neg \exists y.(y \neq x \land \mathsf{color}(n, y))) \land \\
&\forall n \forall m \forall x.(n \neq m \land \mathsf{color}(n, x) \land \mathsf{color}(m, x) \rightarrow \\
&(\mathsf{edge}(n, m) \bigvee_{i=1}^{|N|} (\exists n_1, .., n_i.(\mathsf{edge}(n, n_1) \bigwedge_{j=1}^{i-1} \mathsf{edge}(x_j, x_j + 1) \land \mathsf{edge}(n_i, m)))))
\end{aligned}$$

*

Exercise 3.18.

Minesweeper is a single-player computer game invented by Robert Donner in 1989. The object of the game is to clear a minefield without detonating a mine. The game screen consists of a rectangular field of squares. Each square can be cleared, or uncovered, by clicking on it. If a square that contains a mine is clicked, the game is over. If the square does not contain a mine, one of two things can happen: (1) A number between 1 and 8 appears indicating the amount of adjacent (including diagonally-adjacent) squares containing mines, or (2) no number appears; in which case there are no mines in the adjacent cells. An example of game situation is provided in the following figure:
Provide a first order language that allows to formalize the knowledge of a player in a game state. In such a language you should be able to formalize the following knowledge:

1. *there are exactly* n *mines in the minefield*

2. *if a cell contains the number 1, then there is exactly one mine in the adjacent cells.*

Figure 3.1: An example of a state in the Mines game

3. *show by means of deduction that there must be a mine in the position (3,3) of the game state of picture 3.1.*

Suggestion: define the predicate $Adj(x, y)$ to formalize the fact that two cells x and y are adjacent

Solution.

Language

1. *A unary predicate* mine*, where* $\mathsf{mine}(x)$ *means that the cell* x*contains a mine*
2. *A binary predicate* adj*, where* $\mathsf{adj}(x, y)$ *means that the cell* x *is adjacent to the cell* y
3. *A binary predicate* contains*, where* $\mathsf{contains}(x, n)$ *means that the cell* x *contains the number* n

Axioms

1. *There are exactly* n *mines in the game.*

$$\exists x_1, ..\exists x_n \left(\bigwedge_{i=1}^{n} \mathsf{mine}(x_i) \wedge \forall y \left(\mathsf{mine}(y) \rightarrow \bigvee_{i=1}^{n} y = x_i \right) \right)$$

2. *If a cell contains the number 1, then there is exactly one mine in the adjacent cells.*

   $\forall x. (\mathsf{contains}(x,1) \rightarrow \exists z. (\mathsf{adj}(x,z) \wedge \mathsf{mine}(z) \wedge \forall y. (\mathsf{adj}(x,y) \wedge \mathsf{mine}(y) \rightarrow y = z)))$

3. *Show by means of deduction that there must be a mine in the position (3,3)*
   from Picture 3.1 we have:

   a. $\mathsf{contains}((2,2),1)$

   b. $\neg\mathsf{mine}((1,1)) \wedge \neg\mathsf{mine}((1,2)) \wedge \neg\mathsf{mine}((1,3))$

   c. $\neg\mathsf{mine}((2,1)) \wedge \neg\mathsf{mine}((2,2)) \wedge \neg\mathsf{mine}((2,3))$

   d. $\neg\mathsf{mine}((3,1)) \wedge \neg\mathsf{mine}((3,2))$

   we can deduce:

   e. $\exists z. (\mathsf{adj}((2,2),z) \wedge \mathsf{mine}(z) \wedge \forall y. (\mathsf{adj}((2,2),y) \wedge \mathsf{mine}(y) \rightarrow y = z))$ $\rightsquigarrow$ *from a. and axiom 2*

   f. $\mathsf{mine}((1,1)) \vee \mathsf{mine}((1,2)) \vee \mathsf{mine}((1,3)) \vee \mathsf{mine}((2,1)) \vee \mathsf{mine}((2,2)) \vee \mathsf{mine}((2,3)) \vee \mathsf{mine}((3,1)) \vee \mathsf{mine}((3,2)) \vee \mathsf{mine}((3,3))$ $\rightsquigarrow$ *from e.*

   g. $\mathsf{mine}((3,3))$ $\rightsquigarrow$ *from b.,c.,d. and f.*

*

Exercise 3.19. ☁ ✍

Formalize in first order logic the train connections in Italy. Provide a language that allows to express the fact that a town is directly connected (no intermediate train stops) with another town, by a type of train (e.g., intercity, regional, interregional). Formalize the following facts by means of axioms:

1. *There is no direct connection from Rome to Trento*

2. *There is an intercity from Rome to Trento that stops in Firenze, Bologna and Verona.*

3. *Regional trains connect towns in the same region*

4. *Intercity trains don't stops in small towns.*

Solution. *We define the language as follows*

Constants *$RM, FI, BO, VR, TN, \ldots$ are identifiers of the towns of Roma, Firenze, Bologna, Verona, Trento, ... and InterCity, Regional, ... are the identifiers of the type of trains*

Predicates *Train with arity equal to 1, where $Train(x)$ means x is a train Town with arity equal to 1, where $Train(x)$ means x is a town SmallTown with arity equal to 1, where $Train(x)$ means x is a small town TrainType with arity equal to 2, where $TrainType(x, y)$ means that the train x is of type y. IsInRegion with arity equal to 2, where $IsInRegion(x, y)$ means that the town x is in region y. DirectConn with arity equal to 3, where $DirectConn(x, y, z)$ means that the train x directly connects (with no intermediated stops) the towns y and z.*

Background axioms *With these set of axioms we have to formalize some background knowledge which is necessary to make the formalization more adeguate*

1. *a train has exactly one train type;*

$$\forall x(Train(x) \rightarrow \exists y(TrainType(x, y))) \wedge \forall xyz(TrainType(x, y) \wedge TrainType(x, z) \rightarrow y = z) \tag{3.1}$$

2. *Intercity type is different from regional type:*

$$\neg(InterCity = Regional) \quad (\text{also written as } InterCity \neq Regional) \tag{3.2}$$

3. *A town is associated to exactly one region*

$$\forall x(Town(x) \rightarrow \exists y(IsInRegion(x, y))) \wedge \forall xyz(IsInRegion(x, y) \wedge IsInRegion(x, z) \rightarrow y = z) \tag{3.3}$$

4. *small towns are towns:*

$$\forall x(\textit{SmallTown}(x) \to \textit{Town}(x)) \tag{3.4}$$

5. *if a town a is connected to a town b. b is also connected to a town a.*

$$\forall xy(\exists z\textit{DirectConn}(z,x,y) \to \exists z\textit{DirectConn}(z,y,x)) \tag{3.5}$$

Specific axioms *The axioms that formalizes the specific situation described in the exercise are the following:*

1. *There is no direct connection from Rome to Trento*

$$\neg\exists x\textit{DirectConn}(x,\textit{RM},\textit{TN})$$

2. *There is an intercity from Rome to Trento that stops in Firenze, Bologna and Verona.*

$$\exists x(\textit{DirectConn}(x,\textit{RM},\textit{FI}) \land \textit{DirectConn}(x,\textit{FI},\textit{BO}) \land \textit{DirectConn}(x,\textit{BO},\textit{VR}) \land \textit{DirectConn}(x,\textit{VR},TN) \land \textit{TrainType}(x,\textit{InterCity}))$$

3. *Regional trains connect towns in the same region*

$$\forall xyz(\textit{TrainType}(x,\textit{Regional}) \to (\textit{DirectConn}(x,y,z) \to \exists w(\textit{IsInRegion}(y,w) \land \textit{IsInRegion}(z,w))))$$

4. *Intercity trains don't stops in small towns.*

$$\forall xyz(\textit{DirectConn}(x,y,z) \land \textit{TrainType}(x,\textit{InterCity}) \to \neg\textit{SmallTown}(y) \land \neg\textit{SmallTown}(y))$$

*

Exercise 3.20. ☁ ✍

The game of Draughts is played on a standard Chess board 64 black and white chequered squares. Each player has 12 pieces (men) normally in the form of fat round counters. One player has black men and the other has white men. When starting, each player's men are placed on the 12 black squares nearest to that player (see Figure 3.20). The white squares are not used at all in the game

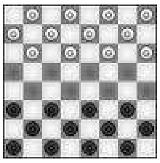


Figure 3.2: Starting position on a 8x8 Draughts board.

- the men only move diagonally and so stay on the black squares throughout. Black always plays first.

Players take turns to move a man of their own colour. There are fundamentally 4 types of move: the ordinary move of a man, the ordinary move of a king, the capturing move of a man and the capturing move of a king.

An ordinary move of a man is its transfer diagonally forward left or right from one square to an immediately neighbouring vacant square. When a man reaches the farthest row forward (the king-row or crownhead) it becomes a king: another piece of the same shade is placed on top of the piece in order to distinguish it from an ordinary man.

An ordinary move of a king is from one square diagonally forward or backward,left or right, to an immediately neighbouring vacant square.

Whenever a piece (man or king) has an opponent's piece adjacent to it and the square immediately beyond the opponent's piece is vacant, the opponent's piece can be captured. If the player has the opportunity to capture one or more of the opponent's pieces, then the player must do so. A piece is taken by simply hopping over it into the vacant square beyond and removing it from the board. Unlike an ordinary move, a capturing move can consist of several such hops - if a piece takes an opponent's piece and the new position allows it to take another piece, then it must do so straight away.

Kings are allowed to move and capture diagonally forwards and backwards and are consequently more powerful and valuable than ordinary men. However, ordinary men can capture Kings.

The game is won by the player who first manages to take all his opponent's pieces or renders them unable to move.

For each of the following conditions on Draughts game write the corresponding axioms, using an appropriate first order logic language.

1. *Each piece is either white or black.*
2. *Each piece is either a king or a man.*
3. *White squares are always empty (always: in each instant of the game).*
4. *In each instant of the game, black squares are either empty or contain a piece.*
5. *At the beginning of the game (instant zero) there are 12 white and 12 black men on the board.*
6. *Whenever a black man captures a white man, in the next instant of the game there is a white man less (and vice-versa).*
7. *If a piece in square* x *captures a piece in square* y *hopping over it into the vacant square* z*, then in the next instant of the game the square* z *contains the piece that moved while squares* x *and* y *are empty.*

Solution.

Language *Predicates:*

- $square(x)$*:* "x is a square"
- $piece(x)$*:* "x is a piece"
- $white(x)$*:* "x is white"
- $black(x)$*:* "x is black"
- $man(x)$*:* "x is a man"
- $king(x)$*:* "x is a king"
- $empty(x, t)$*:* "square x is empty at time t"
- $contain(x, y, t)$*:* "square x contains piece y at time t"
- $capture(x, y, t)$*:* "piece x captures piece y at time t"

Functions:

- $numW(t)$*:* "returns the number of white pieces at time t"
- $numB(t)$*:* "returns the number of black pieces at time t"

Axioms

1. "Each piece is either white or black."

$$\forall x.(piece(x) \rightarrow (white(x) \Leftrightarrow \neg black(x))) \tag{3.6}$$

2. "Each piece is either a king or a man."

$$\forall x.(piece(x) \rightarrow (man(x) \Leftrightarrow \neg king(x))) \tag{3.7}$$

3. "White squares are always empty"

$$\forall x.(square(x) \wedge white(x) \rightarrow \forall t.empty(x,t)) \tag{3.8}$$

4. "In each instant of the game, black squares are either empty or contain a piece."

$$\forall x.(square(x) \wedge black(x) \rightarrow \forall t.(empty(x,t) \Leftrightarrow \neg\exists y.contain(x,y,t))) \tag{3.9}$$

5. "At the beginning of the game (instant zero) there are 12 white and 12 black men on the board."

$$numW(0) = 12 \wedge numB(0) = 12 \tag{3.10}$$

6. "Whenever a black man captures a white man, in the next instant of the game there is a white man less (and vice-versa)."

$$\forall x,y,t.(piece(x) \wedge black(x) \wedge piece(y) \wedge white(y) \wedge capture(x,y,t) \wedge numW(t) = n \\ \rightarrow numW(t+1) = n-1)$$

7. "If a piece in square x captures a piece in square y hopping over it into the vacant square z, then in the next instant of the game the square z contains the piece that moved while squares x and y are empty."

$$\forall p_1,p_2,x,y,z,t.(piece(p_1) \wedge piece(p_2) \wedge square(x) \wedge square(y) \wedge square(z) \wedge empty(z,t) \\ \wedge contain(x,p_1,t) \wedge contain(y,p_2,t) \wedge capture(p_1,p_2,t) \\ \rightarrow empty(x,t+1) \wedge empty(y,t+1) \wedge contain(z,p_1,t+1))$$

*

"The rule is jam tomorrow and jam yesterday but never jam to-day" the Queen said.
"It must come sometimes to "jam to-day,""Alice objected.
"No it can't," said the Queen. "It's jam every other day; to-day isn't any other day, you know."
"I don't understand you," said Alice. "It's dreadfully confusing."

Through the Looking Glass.
LEWIS CARROLL

Chapter 4

Modal Logic

4.1 Basic Concepts

Exercise 4.1.
Say whether the following strings of symbols are well formed modal formulas on $P = \{p, q\}$

1. $\Box \to p$
2. $\Box p \to p$
3. $\Box p \to \Box\Box p$
4. $\Box\Diamond q \wedge \bot\Diamond$
5. $\Box p \to \Diamond p$
6. $\Diamond\top$
7. $p \to \Box\Diamond p$

Solution.

Well formed formulas: 2., 3., 5., 6. and 7.

*

Exercise 4.2.

Say whether the following strings of symbols are well formed modal formulas on $P = \{p, q\}$

1. $\Box\Diamond\Box q$
2. $\Box\Diamond q\Box$
3. $\Diamond p \rightarrow \Box\Diamond p$
4. $(\top \wedge p) \rightarrow \Box q$
5. $\Diamond(p \vee q) \vee (\Diamond p \vee \Diamond q)$
6. $\Box\Diamond q \vee (\Box p \rightarrow \Diamond p(q))$

Solution.

Well formed formulas: 1., 3., 4., and 5.

*

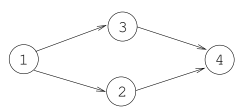
Exercise 4.3.

Let the kripke frame $\mathcal{F} = (W, R)$ *given by*

$$\mathcal{F} = (\{1, 2, 3, 4\}, \{(1, 2), (1, 3), (2, 4), (3, 4)\})$$

Depict the labeled graph corresponding to $\mathcal{F}$.

Solution.



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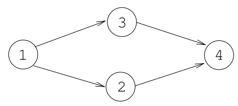
Exercise 4.4.

Let the kripke frame $\mathcal{F} = (W, R)$ *given by*

$$\mathcal{F} = (\{1, 2, 3, 4\}, \{(1, 2), (1, 3), (2, 4), (3, 4)\})$$

Depict the labeled graph corresponding to $\mathcal{F}$*.*

Solution.



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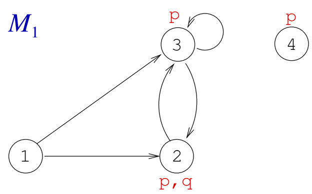
Exercise 4.5.

Suppose $\mathcal{M}_1 = ((W, R), \mathcal{I})$ *is the Kripke model depicted by:*



write down some formulae ϕ *such that* $\mathcal{M}_1, 1 \models \phi$*.*

*

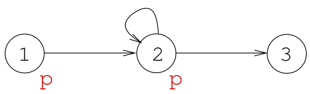
Exercise 4.6.

Consider the following model $\mathcal{M} = ((W, R), \mathcal{I})$ *for the basic modal language of* $P = \{p\}$*:*

$W = \{1,2,3\}$, $R = \{(1,2),(2,2),(2,3)\}$, $\mathcal{I}(p) = \{1,2\}$.
Draw it as a labelled graph and then verify which of the following holds:

1. $\mathcal{M}, 1 \models p$
2. $\mathcal{M}, 2 \models \Diamond p$
3. $\mathcal{M}, 3 \models \Box p$
4. $\mathcal{M}, 1 \models \Box\Box p$
5. $\mathcal{M}, 1 \models \Box\Diamond p$
6. $\mathcal{M}, 1 \models \Diamond\neg p$
7. $\mathcal{M}, 2 \models \Diamond\neg p$
8. $\mathcal{M}, 2 \models (p \to p) \to \Diamond(p \to p)$
9. $\mathcal{M}, 3 \models (p \to p) \to \Diamond(p \to p)$

Solution.



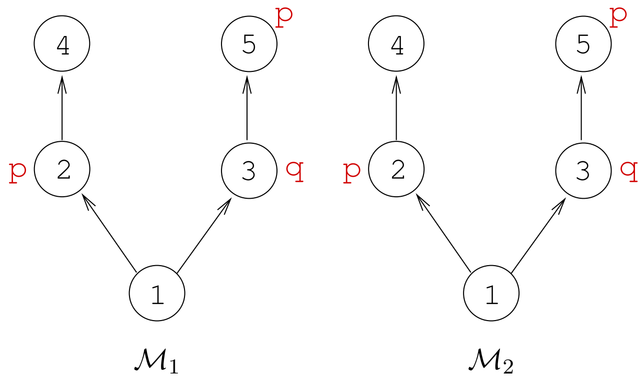
1. ,2. ,3. ,5. ,7. and 8. hold.

*

Exercise 4.7.
Determine whether the following formulas are valid in the lowermost worlds of the two Kripke models below:

1. $\Box p$
2. $\Box q$
3. $\Box p \land q$
4. $\Box\Box\Box\bot$

5. $\Diamond(q \wedge \Diamond(p \wedge \neg q))$



Solution.

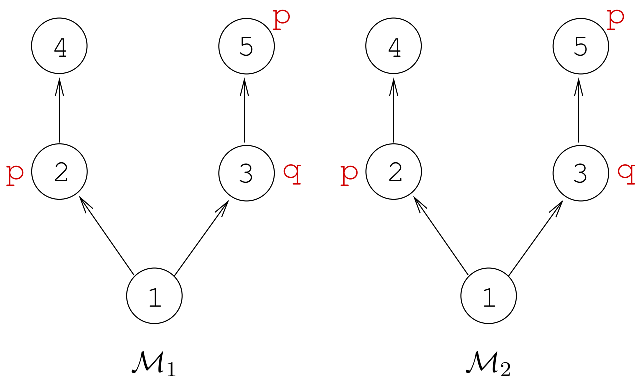
1. $\mathcal{M}_1, 1 \not\models \Box p$ *and* $\mathcal{M}_2, 1 \not\models \Box p$
2. $\mathcal{M}_1, 1 \not\models \Box q$ *and* $\mathcal{M}_2, 1 \models \Box q$
3. $\mathcal{M}_1, 1 \not\models \Box p \wedge q$ *and* $\mathcal{M}_2, 1 \not\models \Box p \wedge q$
4. $\mathcal{M}_1, 1 \models \Box\Box\Box\bot$ *and* $\mathcal{M}_2, 1 \not\models \Box\Box\Box\bot$
5. $\mathcal{M}_1, 1 \models \Diamond(q \wedge \Diamond(p \wedge \neg q))$ $\mathcal{M}_2, 1 \not\models \Diamond(q \wedge \Diamond(p \wedge \neg q))$

*

Exercise 4.8.

Determine in which worlds, of the two Kripke models below, are valid the following formulas:

1. $\Box p$
2. $\Box q$
3. $\Box p \wedge q$
4. $\Box\Box\Box\bot$
5. $\Diamond(q \wedge \Diamond(p \wedge \neg q))$



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Exercise 4.9.

Consider the following model $\mathcal{M}_2 = ((W_2, R_2), \mathcal{I}_2)$ *for the basic modal language of* $P = \{p\}$*:*

$W_2 = \{1, 2\}$, $R_2 = \{(1,2), (2,1)\}$, $\mathcal{I}(p) = \{2\}$.

Draw it as a labelled graph and then verify which of the following holds:

1. $\mathcal{M}_2, 1 \models p \wedge \neg p$
2. $\mathcal{M}_2, 1 \models p \rightarrow p$
3. $\mathcal{M}_2, 2 \models p \wedge \Diamond\neg p$
4. $\mathcal{M}_2, 1 \models p \vee \Diamond\neg p$
5. $\mathcal{M}_2, 1 \models \Box p$
6. $\mathcal{M}_2, 1 \models \Box\Box p$
7. $\mathcal{M}_2, 1 \models \Box\Diamond p$
8. $\mathcal{M}_2, 2 \models \Box\Diamond p$
9. $\mathcal{M}_2, 2 \models \Diamond\Box\neg p$
10. $\mathcal{M}_2, 1 \models \Diamond\Box\neg p \rightarrow \neg p$
11. $\mathcal{M}_2, 2 \models \Diamond\Box\neg p \rightarrow \neg p$

*

Exercise 4.10. ☼

Consider the following model $\mathcal{M}_3 = ((W_3, R_3), \mathcal{I}_3)$ for the basic modal language of $P = \{p, q\}$:
$W_3 = \{1,2,3,4\}$, $R_2 = \{(1,2),(2,3),(3,1),(3,4),(4,2)\}$, $\mathcal{I}(p) = \{1,3\}$, $\mathcal{I}(q) = \{1,2\}$.
Draw it as a labelled graph and then verify which of the following holds:

1. $\mathcal{M}_3, 1 \models \Box q$
2. $\mathcal{M}_3, 1 \models \Box\neg(p \to q)$
3. $\mathcal{M}_3, 3 \models \Box((p \wedge q) \vee (\neg p \wedge \neg q))$
4. $\mathcal{M}_3, 1 \models \Diamond\Box p$
5. $\mathcal{M}_3, 1 \models \Diamond p \wedge \Diamond q$
6. $\mathcal{M}_3, 1 \models \Diamond p \vee \Diamond q$
7. $\mathcal{M}_3, 3 \models \Diamond(p \vee \neg q)$
8. $\mathcal{M}_3, 3 \models \Diamond p \wedge \Diamond\neg q$

*

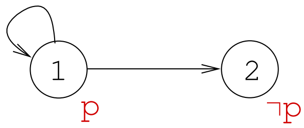
4.2 Satisfiability and Validity

Exercise 4.11. ☼ ✍

Is $\Diamond p \wedge \Diamond\neg p$ satisfiable?

Solution.

It's sufficient to find a model $\mathcal{M}$ *and a world* w *in* $\mathcal{M}$ *such that* $\mathcal{M}, w \models \Diamond p \wedge \Diamond \neg p$*:*



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Exercise 4.12. ☁ ✍

Is $\Diamond p \wedge \Box \neg p$ *satisfiable?*

Solution.

$\mathcal{M}, w \models \Diamond p \wedge \Box \neg p$

iff $\mathcal{M}, w \models \Diamond p$ *and* $\mathcal{M}, w \models \Box \neg p$

iff *for some world* $v \in W$*,* $(w, v) \in R$ *and* $\mathcal{M}, v \models p$*, and* $\mathcal{M}, w \models \Box \neg p$

iff *for some world* $v \in W$*,* $(w, v) \in R$ *and* $\mathcal{M}, v \models p$*, and for every world* $u \in W$*,* $(w, u) \in R$ *implies* $\mathcal{M}, u \models \neg p$

iff *for some world* $v \in W$*,* $(w, v) \in R$ *and* $v \in \mathcal{I}(p)$*, and for every world* $u \in W$*,* $(w, u) \in R$ *implies* $u \notin \mathcal{I}(p)$

iff *for some world* $v \in W$*,* $(w, v) \in R$ *and* $v \in \mathcal{I}(p)$*, and* $(w, v) \in R$ *implies* $v \notin \mathcal{I}(p)$

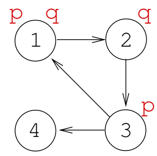
iff *for some world* $v \in W$*,* $(w, v) \in R$ *and* $v \in \mathcal{I}(p)$ *and* $v \notin \mathcal{I}(p)$

We can conclude that $\Diamond p \wedge \Box \neg p$ *is* **unsatisfiable**.

*

Exercise 4.13. ☼ ✍

Consider the simple Kripke model below:



Decide whether the following formulas are valid in the model:

1. $\Diamond\Box p \vee \Diamond\Diamond\Box p$

2. $\Box p \to \neg p$

3. $\Diamond(p \vee \neg p) \to (p \vee \Diamond\neg q)$

4. $\Box\Box\Diamond q$

Solution.

1. $M, 2 \not\models \Diamond\Box p \vee \Diamond\Diamond\Box p$,
   thus $M \not\models \Diamond\Box p \vee \Diamond\Diamond\Box p$

2. $M, 1 \models \Box p \to \neg p$,
   $M, 2 \models \Box p \to \neg p$,
   $M, 3 \models \Box p \to \neg p$, *and* $M, 4 \models \Box p \to \neg p$
   thus $M \models \Box p \to \neg p$

3. $M, 1 \models \Diamond(p \vee \neg p) \to (p \vee \Diamond\neg q)$,
   $M, 2 \models \Diamond(p \vee \neg p) \to (p \vee \Diamond\neg q)$,
   $M, 3 \models \Diamond(p \vee \neg p) \to (p \vee \Diamond\neg q)$,
   and $M, 4 \models \Diamond(p \vee \neg p) \to (p \vee \Diamond\neg q)$
   thus $M \models \Diamond(p \vee \neg p) \to (p \vee \Diamond\neg q)$

4. $M, 2 \not\models \Box\Box\Diamond q$,
   thus $M \not\models \Box\Box\Diamond q$

✻

Exercise 4.14.
Prove that $\Diamond p \leftrightarrow \neg\Box\neg p$ *is valid in any Kripke model.*

Solution. *Suppose $\mathcal{M} = ((W, R), \mathcal{I})$ is an arbitrary Kripke model and w is an arbitrary world in $\mathcal{M}$.*

$\mathcal{M}, w \models \neg\Box\neg p$

iff   *not* $\mathcal{M}, w \models \Box\neg p$

iff   *not for every* v, $w, v \in R$ *implies* $\mathcal{M}, w \models \neg p$

iff   *for some* v, $w, v \in R$ *and not* $\mathcal{M}, w \models \neg p$

iff   *for some* v, $w, v \in R$ *and* $\mathcal{M}, w \models \neg\neg p$

iff   *for some* v, $w, v \in R$ *and* $\mathcal{M}, w \models p$

iff $\mathcal{M}, w \models \Diamond p$

*

Exercise 4.15.

Prove that the following facts hold:

1. $\models p \wedge (p \to q) \to q$ *(*modus ponens*)*

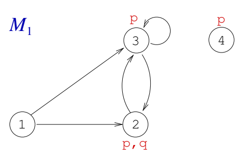
2. *if* $\models p$ *then* $\models \Box p$ *(*necessitation*)*

*

Exercise 4.16.

Prove that $\Box p \to \Diamond p$ (D) is not valid in the class of all frames.

Solution. *We need to find a model and a world in the model which falsify the formula.*



$\mathcal{M}_1, 4 \not\models \Box p \to \Diamond p$

*

Exercise 4.17.

Prove that the following formulae are not valid in the class of all frames.

1. $\Box p \to p$
2. $p \to \Box\Diamond p$
3. $\Box p \to \Box\Box p$
4. $\Diamond p \to \Box\Diamond p$

*

Exercise 4.18.

For each of the following formulas, show that it is valid (i.e., true in all models) or find a counterexample

1. $\Box A \land \Diamond B \to \Diamond(A \land B)$
2. $A \to \Box\Diamond A$
3. $\Box(A \lor \Box B) \to \Box A \lor \Box\Box B$

Solution.

1. $\Box A \land \Diamond B \to \Diamond(A \land B)$

$$\begin{aligned}
M, w \models \Box A \land \Diamond B &\Leftrightarrow M, w \models \Box A \textit{ and } M, w \models \Diamond B \\
&\Leftrightarrow \textit{for all } w' \textit{ with } wRw'\textit{, } M, w' \models A \textit{ and} \\
&\qquad \textit{there is a } w_0 \textit{ with } wRw_0\textit{, and } M, w_0 \models B \\
&\Rightarrow \textit{there is a } w_0 \textit{ with } wRw_0\textit{, and } M, w_0 \models B \textit{ and } M, w_0 \models A \\
&\Leftrightarrow \textit{there is a } w_0 \textit{ with } wRw_0\textit{, and } M, w_0 \models A \land B \\
&\Leftrightarrow M, w \models \Diamond A \land B
\end{aligned}$$

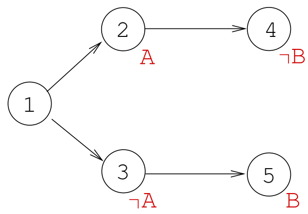
2. $A \to \Box\Diamond A$

   Counterexample:

$\mathcal{M}, 1 \not\models A \to \Box\Diamond A$

3. $\Box(A \vee \Box B) \to \Box A \vee \Box\Box B$

   Counterexample:



$\mathcal{M}, 1 \not\models \Box(A \vee \Box B) \to \Box A \vee \Box\Box B$

∗

Exercise 4.19.

For each of the following formulas, either prove that it is valid or find a Kripke model $\langle W, R, I\rangle$ and a $w \in W$ that does not satisfy it.

1. $\Diamond\Box A \to \Box\Box A$

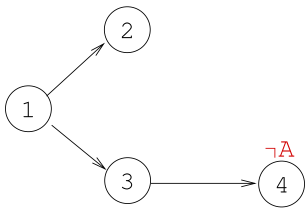
2. $\Diamond A \wedge (\Box B \vee \Box C) \to \Diamond(A \wedge (B \vee C))$

3. $\Box\Box A \to \Box A$

Solution.

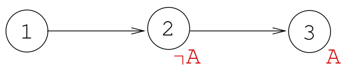
1. $\Diamond\Box A \to \Box\Box A$

   Counterexample:



2. $\Diamond A \wedge (\Box B \vee \Box C) \to \Diamond(A \wedge (B \vee C))$
This formula is valid as, $w_0 \models \Diamond A \wedge (\Box B \vee \Box C)$ *implies that there is a world* w_1 *accessible from* w_0 *such that* $w_1 \models A$*. Suppose* $w_0 \models \Box B$ *then* $w_1 \models B$ *and therefore* $w_1 \models A \vee B$*. If, instead* $w_0 \models \Box C$*, then* $w_1 \models C$ *and therefore* $w_1 \models A \wedge C$*. In both cases* $w_1 \models A \wedge (B \vee C)$*. Which implies that* $w_0 \models \Diamond(A \wedge (B \vee C))$*.*

3. $\Box\Box A \to \Box A$
Counterexample:



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Exercise 4.20.

For each of the following formulas either prove that it is valid or find a counterexample. Note that if your attempts to produce a falsifying model always end in incoherent pictures, it may be because the formula is valid.

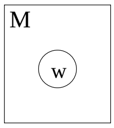
1. $\Box\Diamond A \to \Diamond\Diamond A$

2. $\Diamond(\Box A \wedge \Diamond B) \to \Diamond\Diamond \mathrm{T}$

3. $\neg\Diamond\Box A \to \Diamond\Diamond\neg A$

Solution.

1. *Countermodel:*

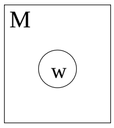


$M, w \not\models \Box\Diamond A \to \Diamond\Diamond A$.

2. $\Diamond(\Box A \wedge \Diamond B) \to \Diamond\Diamond \mathbf{T}$ *is valid.*
   We prove that for all models M *and for all worlds* w*,* $M, w \models \Diamond(\Box A \wedge \Diamond B) \to \Diamond\Diamond \mathbf{T}$.

$$\begin{aligned}
M, w \models \Diamond(\Box A \wedge \Diamond B) &\Leftrightarrow \text{there exists a } v \in W \text{ s.t. } wRv \text{ and } M, v \models \Box A \wedge \Diamond B \\
&\Leftrightarrow \text{there exists a } v \in W \text{ s.t. } wRv \text{ and } M, v \models \Box A \text{ and } M, v \models \Diamond B \\
&\Leftrightarrow \text{there exists a } v \in W \text{ s.t. } wRv \text{ and } M, v \models \Box A \\
&\quad\ \text{and there exists a } v' \text{ s.t. } vRv' \text{ and } M, v' \models B \\
&\Rightarrow \text{there exist } v, v' \in W \text{ s.t. } wRv \text{ and } vRv' \\
&\Leftrightarrow M, w \models \Diamond\Diamond \mathbf{T}
\end{aligned}$$

3. *Countermodel:*



$M, w \not\models \neg\Diamond\Box A \to \Diamond\Diamond\neg A$.

*

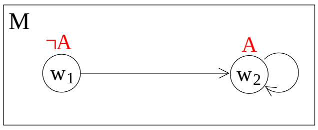
Exercise 4.21.
For each of the following formulas, either prove that it is valid or find a Kripke model $\langle W, R, I\rangle$ *and a* $w \in W$ *that does not satisfy it.*

1. $\Box\Diamond A \to A$
2. $(\Diamond A \to \Box B) \to (\Box A \to \Box B)$

Solution.

1. $\Box\Diamond A \to A$

   Counterexample:



   $M, w_1 \not\models \Box\Diamond A \to A$.

2. $(\Diamond A \to \Box B) \to (\Box A \to \Box B)$

   We prove that for all models M *and for all worlds* w, $M, w \models (\Diamond A \to \Box B) \to (\Box A \to \Box B)$. *Consider any model* M *and any world* w. *We have two cases:*

   - *case 1:* $M, w \not\models \Diamond A \to \Box B$

     $M, w \not\models \Diamond A \to \Box B \implies$ *by def. of* $\to$, $M, w \models (\Diamond A \to \Box B) \to (\Box A \to \Box B)$.

   - *case 2:* $M, w \models \Diamond A \to \Box B$

     then we have two cases:

     – *case 2.1:* $M, w \not\models \Diamond A$

$$\begin{aligned} M, w \not\models \Diamond A &\implies M, w \not\models \Box A \\ &\implies M, w \models \Box A \to \Box B \\ &\implies M, w \models (\Diamond A \to \Box B) \to (\Box A \to \Box B) \end{aligned}$$

     – *case 2.2:* $M, w \models \Diamond A$

$$\begin{aligned} M, w \models \Diamond A &\implies \text{since } M, w \models \Diamond A \to \Box B,\ M, w \models \Box B \\ &\implies \text{by def of } \to,\ M, w \models \Box A \to \Box B \\ &\implies M, w \models (\Diamond A \to \Box B) \to (\Box A \to \Box B) \end{aligned}$$

*

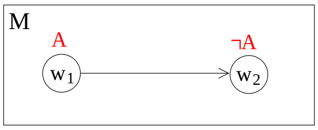
Exercise 4.22. ☁ ✍

For each of the following formulas, either prove that it is valid or find a Kripke model $\langle W, R, I\rangle$ *and a* $w \in W$ *that does not satisfy it.*

1. $A \to \Box A$
2. $(\neg\Diamond A \land \Diamond B) \to \Diamond(\neg A \land B)$
3. $\Box\Diamond A \to \Diamond\Box A$

Solution.

1. *Countermodel:*

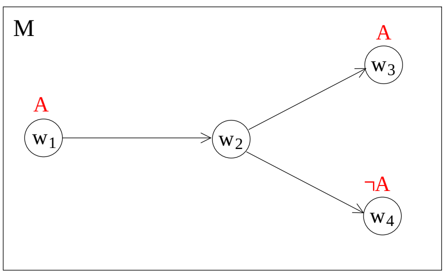


$M, w_1 \not\models A \to \Box A.$

2. *We prove that for all models* M *and for all worlds* w*,* $M, w \models (\neg\Diamond A \land \Diamond B) \to \Diamond(\neg A \land B)$.

$$\begin{aligned}
M, w \models \neg\Diamond A \land \Diamond B \quad &\Leftrightarrow \quad M, w \models \neg\Diamond A \text{ and } M, w \models \Diamond B\\
&\Leftrightarrow \quad M, w \not\models \Diamond A \text{ and } M, w \models \Diamond B\\
&\Leftrightarrow \quad \text{for all } w', wRw' \text{ implies } M, w' \not\models A \text{ and}\\
& \quad \text{there is a } w_0 \text{ with } wRw_0, \text{ and } M, w_0 \models B\\
&\Rightarrow \quad \text{there is a } w_0 \text{ with } wRw_0 \text{ s.t. } M, w_0 \not\models A \text{ and } M, w_0 \models B\\
&\Leftrightarrow \quad \text{there is a } w_0 \text{ with } wRw_0 \text{ s.t. } M, w_0 \models \neg A \text{ and } M, w_0 \models B\\
&\Leftrightarrow \quad M, w \models \Diamond 9(\neg A \land B)
\end{aligned}$$

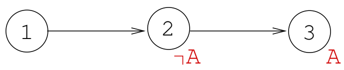
3. *Countermodel:*



$M, w_1 \not\models \Box\Diamond A \to \Diamond\Box A$

4. $\Box\Box A \to \Box A$

   Counterexample:



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Exercise 4.23.

Prove that the modal logic formula

$$\Box\phi \to \phi$$

is valid in the class of reflexive Kripke frames.

Solution. *Suppose $\mathcal{F} = (W, R)$ is a frame with reflexive R. Let $\mathcal{I}$ and let $\mathcal{M} = ((W, R), \mathcal{I})$. We need to show that for every world $v \in W$ and for every formula ϕ,*

$$\mathcal{M}, v \models \Box\phi \to \phi$$

Let v be an arbitrary element of W and ϕ an arbitrary modal formula. We distinguish two cases:

1. $\mathcal{M}, v \not\models \Box\phi$

   Then by def. of $\to$, $\quad \mathcal{M}, v \models \Box\phi \to \phi$

2. $\mathcal{M}, v \models \Box\phi$

   Then by def. of $\Box$

   for every $u \in W$, if $(v, u) \in R$ then $\mathcal{M}, u \models \phi$.

   R is reflexive, hence $(v, v) \in R$.

   Therefore $\mathcal{M}, v \models \phi$.

   So, $\mathcal{M}, v \models \Box\phi \to \phi$.

*

Exercise 4.24. 🌧 ✍

Prove that if, for every formula ϕ, the modal logic formula

$$\Box\phi \to \phi$$

is valid in a Kripke frame $\mathcal{F} = (W, R)$, then R is reflexive.

Solution. *Assume that $\Box\phi \to \phi$ is valid in $\mathcal{F} = (W, R)$, for every ϕ.*

Then, for every $\mathcal{I}$ and for any world $w \in W$ we have

$$\textit{if } \mathcal{M}, w \models \Box\phi \quad \textit{then } \mathcal{M}, w \models \phi.$$

We need to show that R is reflexive.

Assume R is not reflexive.
$\Longrightarrow$ *there is a world $w \in W$ s.t. $(w, w) \notin R$*
Let $\mathcal{I}$ be an interpretation function s.t. $(\mathcal{F}, \mathcal{I}), w \not\models \phi$ and for all $v \in W$, $v \neq w$, $(\mathcal{F}, \mathcal{I}), v \models \phi$.

There are 2 cases:

1. *w has a successor v (with $v \neq w$)*
2. *w is a dead-end (no successors)*

In both cases $\mathcal{M}, w \models \Box\phi$, but $\mathcal{M}, w \not\models \phi$.

CONTRADICTION!

✻

Exercise 4.25. ☁ ✍

Show that in the frame $\mathcal{F} = (W, R)$ if R is an equivalence relation then the schema $\phi \to \Box\Diamond\phi$ is valid

Solution. *We need to prove that for any model M and any world w, $M, w \models \phi \to \Box\Diamond\phi$.*
Consider two cases:

- *case 1:* $M, w \not\models \phi$

$$M, w \not\models \phi \implies \textit{by def of} \rightarrow, M, w \models \phi \rightarrow \Box\Diamond\phi$$

- *case 2:* $M, w \models \phi$
since R is an equivalence relation, it is simmetric, then we have that for all $w', wRw' \rightarrow w'Rw$

$$\begin{aligned} M, w \models \phi \textit{and for all } w', wRw' \rightarrow w'Rw &\implies \textit{for all } w', wRw' \textit{ implies } M, w' \models \Diamond\phi \\ &\implies M, w \models \Box\Diamond\phi \\ &\implies M, w \models \phi \rightarrow \Box\Diamond\phi. \end{aligned}$$

*

Exercise 4.26.

Show that if a frame $\langle W, R\rangle$ satisfy the schema $\Box\phi \rightarrow \Box\Box\phi$ then R is transitive.

*

Exercise 4.27.

Let $\mathcal{F} = (W, R)$ be a frame.
Prove that the following statements are equivalent:

1 *R is transitive*

2 *$\Box\phi \rightarrow \Box\Box\phi$ is valid in $\mathcal{F}$ (for every ML formula ϕ)*

*

4.3 Modal Logic Formalization

Exercise 4.28.

Provide a modal language and a set of axioms that formalize the **graph coloring problem** *of a graph with at most n nodes, with connection degree $\leq m$, and with less then $k+1$ colors.*

- node degree*: number of adjacent nodes*
- connection degree *of a graph: max among all the degree of its nodes*
- graph coloring problem*: given a non-oriented graph, associate a color to each of its nodes in such a way that no pair of adjacent nodes have the same color.*

Solution.

Language *A set of propositional variables, $C_1, \ldots C_k$ one for each color.*

Axioms

1. *"Each node has at least one color":*

$$\bigvee_{i=1}^{k} C_i$$

2. *"Each node has at most one color":*

$$\bigwedge_{i \neq j=1}^{k} C_i \rightarrow \neg C_j$$

3. *"Each node does not have the same color as an accessible nodes:*

$$\bigwedge_{i=1}^{k} (C_i \rightarrow \Box \neg C_i)$$

4. *"Each node does not have more than m successors":*

$$\bigwedge_{i=1}^{m+1} \Diamond\phi_i \rightarrow \bigvee_{i \neq j=1}^{m+1} \Diamond(\phi_i \wedge \phi_j)$$

*

Exercise 4.29. ☁ ✍

Show how it is possible to represent the railways connections in a country by means of a Kripke frame. First, select the schema you have to impose to capture the following fact: "if there is a direct train connection to go from a to b, then there is also a train connection in the opposite direction

Then, provide a set of axioms to formalize the following statements.

1. *You cannot be at the same time in Roma and Firenze*
2. *There is no direct train connection from Roma to Trento.*
3. *From Rome you can reach Trento with 2 changes.*
4. *At Riva del Garda there is no train station.*

Solution. *If the train direct connections is represented by the relation R of a Kripke frame, and each world is considered as a train stop, then then the condition*

"if there is a direct train connection to go from a to b, then there is also a train connection in the opposite direction"

can be imposed by requiring that R is symmetric. Symmetry of the accessibility relation can be strongly represented by means of the schema

$$\phi \rightarrow \Box\Diamond\phi$$

As far as the other conditions, they can be represented by means axioms, on a language that contains the propositions RM, TN, FI ... (meaning that we are at in Rome, Trento, Firenze, ...).

1. *You cannot be at the same time in Roma and Firenze*

$$RM \rightarrow \neg FI$$

2. *There is no direct train connection from Roma to Trento.*

$$RM \rightarrow \neg\Diamond TN$$

3. *From Rome you can reach Trento with 2 changes.*

$$RM \rightarrow \Diamond\Diamond\Diamond TN$$

4. *At Riva del Garda there is no train station.*

$$RivaDelGarda \rightarrow \Box\bot$$

*