Mathematical Logic

Modal Logics - exercises

Chiara Ghidini

FBK-irst, Trento, Italy

December 16, 2014

Exercises on Modal Logics

Exercise

Prove that the following formulas are valid. I.e., they are valid in every frame.

1. $\Diamond(\phi \vee \psi) \supset (\Diamond\phi \vee \Diamond\psi)$
2. $\Diamond(\phi \wedge \psi) \supset \Diamond\phi$
3. $\Box\phi \wedge \Diamond\psi \supset \Diamond(\phi \wedge \psi)$
4. $\Diamond\phi \supset (\Box\psi \supset \neg\Box\neg\psi)$
5. $\Diamond^n\bot \supset \neg\Box^m\bot$ for $m \leq n$
6. $\Diamond\phi \vee \Diamond\neg\phi \vee \Box\bot$
7. $\Box\phi \vee \Box\neg\phi \vee (\Diamond\phi \wedge \Diamond\neg\phi)$

Exercises on Modal Logics

Solution

1. $\Diamond(\phi \vee \psi) \supset (\Diamond\phi \vee \Diamond\psi)$

   - *If* $\mathcal{M}, w \models \Diamond(\phi \vee \psi)$,
   - *then there is a* w' *accessible from* w *(i.e.,* wRw'*) such that* $\mathcal{M}, w' \models \phi \vee \psi$
   - *which implies that either* $\mathcal{M}, w' \models \phi$ *or* $\mathcal{M}, w' \models \psi$
     - *If* $\mathcal{M}, w' \models \phi$, *then, since* wRw' *we have that* $\mathcal{M}, w \models \Diamond\phi$
     - *If* $\mathcal{M}, w' \models \psi$, *then, since* wRw' *we have that* $\mathcal{M}, w \models \Diamond\psi$
   - *in both cases we have that* $\mathcal{M}, w \models \Diamond\phi \vee \Diamond\psi$.
   - *therefore we conclude that* $\mathcal{M}, w \models \Diamond(\phi \vee \psi) \supset (\Diamond\phi \vee \Diamond\psi)$.

2. $\Diamond(\phi \wedge \psi) \supset \Diamond\phi$

   - *If* $\mathcal{M}, w \models \Diamond(\phi \wedge \psi)$,
   - *then there is a* w' *with* wRw' *such that* $\mathcal{M}, w' \models \phi \wedge \psi$
   - *which implies that* $\mathcal{M}, w' \models \phi$
   - *since* wRw' *we have that* $\mathcal{M}, w \models \Diamond\phi$
   - *therefore we conclude that* $\mathcal{M}, w \models \Diamond(\phi \wedge \psi) \supset \Diamond\phi$

Solution

1. $\Box\phi \wedge \Diamond\psi \supset \Diamond(\phi \wedge \psi)$

   - *If* $\mathcal{M}, w \models \Box\phi \wedge \Diamond\psi$ *then* $\mathcal{M}.w \models \Box\phi$ *and* $\mathcal{M}, w \models \Diamond\psi$.
   - $\mathcal{M}, w \models \Diamond\psi$ *implies that there is a* w' *with* wRw' *such that* $\mathcal{M}, w' \models \psi$.
   - $\mathcal{M}, w \models \Box\psi$ *implies that for all world accessible from* w, *and therefore also for* w', $\mathcal{M}, w' \models \phi$,
   - *this allows to conclude that* $\mathcal{M}, w' \models \phi \wedge \psi$
   - *and therefore, since* wRw', $\mathcal{M}, w \models \Diamond(\phi \wedge \psi)$.

Solution

1. $\Diamond\phi \supset (\Box\psi \supset \neg\Box\neg\psi)$

   - $\mathcal{M}, w \models \Diamond\phi$ *implies that there is a world* w' *with* wRw' *such that* $\mathcal{M}, w' \models \phi$. *(In the proof we will use only the fact that there is a world* w' *accessible from* w*, the fact that* $\mathcal{M}, w' \models \phi$ *is completely irrelevant)*
   - *suppose that* $\mathcal{M}, w \models \Box\psi$,
   - *than since* wRw'*, and for all world accessible from* w*,* ψ *must be true, we have that* $\mathcal{M}, w' \models \psi$,
   - *which implies that* $\mathcal{M}, w' \not\models \neg\psi$
   - *the fact that* w *has an accessible world* w' *with* $\mathcal{M}, w' \not\models \neg\psi$ *implies that* $\mathcal{M}, w \not\models \Box\neg\psi$
   - *which implies that* $\mathcal{M}, w \models \neg\Box\neg\psi$
   - *we can therefore conclude that* $\mathcal{M}, w \models \Box\psi \supset \neg\Box\neg\psi$ *under that assumption that* $\mathcal{M}, w \models \Diamond\phi$
   - *and therefore we conclude that* $\mathcal{M}, w \models \Diamond\phi \supset (\Box\psi \supset \neg\Box\neg\psi)$.

Exercises on Modal Logics

Solution

1. $\Diamond^n\top \supset \neg\Box^m\bot$ *for* $m \leq n$

For $n, m \geq 0$, $\Diamond^n$ *stands for* $\overbrace{\Diamond\ldots\Diamond}^{n\ times}$.

- $\Diamond^1\top$ *is equal to* $\Diamond\top$, *and we have that* $\mathcal{M}, w \models \Diamond^1\top$, *if there is a possible world* w_1 *accessible from* w *(i.e.,* wRw_1*)*
- $\Diamond^2\top$ *is equal to* $\Diamond\Diamond\top$. *therefore* $\mathcal{M}, w \models \Diamond^2\top$ *if there is a world* w_1 *with* wRw_1 *such that* $\mathcal{M}, w_1 \models \Diamond\top$, *which in turn is true if there is a world* w_2 *with* w_1Rw_2.
- *continuing reasoning like above, we have that* $\mathcal{M}, w \models \Diamond^n\top$ *if there are* n *worlds* $w_1, \ldots, w_n$ *such that* wRw_1,w_1Rw_2 $\ldots w_{n-1}Rw_n$, *i.e., if there is a path (it can be also circular) of* n *steps.*

Exercises on Modal Logics

Solution

1. $\Diamond^n \top \supset \neg \Box^m \bot$ for $m \leq n$ (cont'd)

The formula $\Box^m$ stands for $\overbrace{\Box \ldots \Box}^{m \text{ times}}$.

- $\mathcal{M}, w \models \neg \Box^1 \bot$ means that $\mathcal{M}, w \models \neg \Box \bot$,
- which implies that $\mathcal{M}, w \neg \models \Box \bot$. Notice that the only case in which $\mathcal{M}, \models \Box \bot$ is when there is no world accessible from w.
- therefore, $\mathcal{M}, w \not\models \Box \bot$ implies that there iw a world w_1 accessible from w, i.e., wRw'.
- $\mathcal{M}, w \models \neg \Box^2 \bot$ means that $\mathcal{M}, w \not\models \Box^2 \bot$,
- this implies that there is a w_1 accessible from w such that $\mathcal{M}, w_1 \not\models \Box \bot$, which in turns implies that there is a world w_2 accessible form w_1.
- iterating m times the above reasoning we have that $\mathcal{M}, w \models \neg \Box^m \bot$ if there are m worlds $w_1, \ldots, w_m$ with wRw_1, $w_1 Rw_2$, ..., $w_{m-1} Rw_m$. i.e., if there is a path of length m.

Exercises on Modal Logics

Solution

1. $\Diamond^n \phi \supset \neg \Box^m \bot$ *for* $m \leq n$ *(cont'd)*

   Summarizing:

   - $\mathcal{M}, w \models \Diamond^n \top$ *if there is a path of length* n
   - $\mathcal{M}, w \models \neg \Box^m \bot$ *if there is a path of length* m
   - *the fact that* $m \leq n$ *implies that if there is a path of length* n *there is also a path of length* m *(just take the first* m *steps of the path of length* n*)*
   - *which implies that* $\mathcal{M}, w \models \Diamond^n \top \supset \neg \Box^m \bot$ *with* $m \leq n$.

2. $\Diamond \phi \vee \Diamond \neg \phi \vee \Box \bot$

   - *Suppose that* $\mathcal{M}, w \not\models \Box \bot$ *then*
   - *there is a world* w'*, with* wRw'.
   - *we have that either* $\mathcal{M}, w' \models \phi$ *or* $\mathcal{M}, w' \not\models \phi$
   - *in the first case we have that* $\mathcal{M}, w \models \Diamond \phi$
   - *in the second case* $\mathcal{M}, w' \models \neg \phi$ *and therefore* $\mathcal{M}, w \models \Diamond \neg \phi$.
   - *This implies that either* $\Box \bot$ *or* $\Diamond \phi$ *or* $\Diamond \neg \phi$ *is true in* w.
   - *and therefore* $\mathcal{M}, w \models \Diamond \phi \vee \Diamond \neg \phi \vee \Box \bot$.

Exercises on Modal Logics

Solution

1. $\Box\phi \lor \Box\neg\phi \lor (\Diamond\phi \land \Diamond\neg\phi)$

   - *Suppose that* $\mathcal{M}, w \not\models \Box\phi$
   - *This implies that there is a world* w_1 *accessible from* w *such that* $\mathcal{M}, w_1 \not\models \phi$,
   - *this implies that* $\mathcal{M}, w_1 \models \neg\phi$ *and therefore* $\mathcal{M}, w \models \Diamond\neg\phi$
   - *Suppose that* $\mathcal{M}, w \not\models \Box\neg\phi$ *then there is a world* w_2 *accessible from* w *such that* $\mathcal{M}, w_2 \not\models \neg\phi$,
   - *this implies that* $\mathcal{M}, w_2 \models \phi$ *and therefore that* $\mathcal{M}, w \models \Diamond\phi$.
   - *we can conclude that if* $\mathcal{M}, w \not\models \Box\phi$ *and* $\mathcal{M}, w \not\models \Box\neg\phi$, *then* $\mathcal{M}, w \models \Diamond\phi \land \diamond\neg\phi$.
   - *which implies that* $\mathcal{M}, w \models \Box\phi \lor \Box\neg\phi \lor (\Diamond\phi \land \Diamond\neg\phi)$.

Tableau Rules for the Propositional Logic

Expansion rules for propositional connectives

$$\frac{w \models \phi \wedge \psi}{\begin{array}{c} w \models \phi \\ w \models \psi \end{array}} \qquad \frac{w \not\models (\phi \vee \psi)}{\begin{array}{c} w \not\models \phi \\ w \not\models \psi \end{array}} \qquad \frac{w \models \neg\phi}{w \not\models \phi} \qquad \frac{w \not\models \neg\phi}{w \models \phi} \qquad \frac{w \not\models (\phi \supset \psi)}{\begin{array}{c} w \models \phi \\ w \not\models \psi \end{array}}$$

$$\frac{w \models \phi \vee \psi}{w \models \phi \mid w \models \psi} \qquad \frac{w \not\models (\phi \wedge \psi)}{w \not\models \phi \mid w \not\models \psi} \qquad \frac{w \models \phi \supset \psi}{w \not\models \phi \mid w \models \psi}$$

Expansion rules for modal operators

$$\frac{w \models \Box\phi}{w' \models \phi}$$ If wRw' is already in the brench

$$\frac{w \not\models \Box\phi}{\begin{array}{c} wRw' \\ w' \not\models \phi \end{array}}$$ wher w' is new in the brench

$$\frac{w \models \Diamond\phi}{\begin{array}{c} wRw' \\ w' \models \phi \end{array}}$$ wher w' is new in the brench

$$\frac{w \not\models \Diamond\phi}{w' \not\models \phi}$$ If wRw' is already in the brench