Mathematical Logic

Propositional Logic - Syntax and Semantics

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Propositional logic - Intuition

- Propositional logic is the logic of propositions
- a proposition can be true or false in the state of the world.
- the same proposition can be expressed in different ways. E.g.
  - "B. Obama is drinking a bier"
  - " The U.S.A. president is drinking a bier", and
  - "B. Obama si sta facendo una birra"

  express the same proposition.
- The language of propositional logic allows us to express propositions.

Propositional logic language

Definition (Propositional alphabet)

Logical symbols $\neg$, $\wedge$, $\vee$, $\supset$, and $\equiv$

Non logical symbols A set $\mathcal{P}$ of symbols called propositional variables

Separator symbols "(" and ")"

Definition (Well formed formulas (or simply formulas))

- every $P \in \mathcal{P}$ is an atomic formula
- every atomic formula is a formula
- if A and B are formulas then $\neg A$, $A \wedge B$, $A \vee B$ $A \supset B$, e $A \equiv B$ are formulas

Formulas cont'd

Example ((non) formulas)

Formulas	Non formulas
$P \rightarrow Q$	PQ
$P \rightarrow (Q \rightarrow R)$	$(P \rightarrow \wedge((Q \rightarrow R)$
$P \wedge Q \rightarrow R$	$P \wedge Q \rightarrow \neg R \neg$

Reading formulas

Problem

How do we read the formula $P \wedge Q \to R$?
The formula $P \wedge Q \to R$ can be read in two ways:

1. $(P \wedge Q) \to R$
2. $P \wedge (Q \to R)$

Symbol priority

$\neg$ has higher priority, then $\wedge$, $\vee$, $\to$ and $\equiv$. Parenthesis can be used around formulas to stress or change the priority.

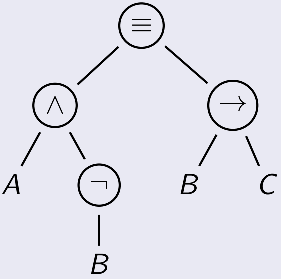
Symbol	Priority
$\neg$	1
$\wedge$	2
$\vee$	3
$\to$	4
$\equiv$	5

Formulas as trees

Tree form of a formula

A formula can be seen as a tree. Leaf nodes are associated to propositional variables, while intermediate (non-leaf) nodes are associated to connectives.

For instance the formula $(A \wedge \neg B) \equiv (B \rightarrow C)$ can be represented as the tree



Subformulas

Definition

(Proper) Subformula

- A is a subformula of itself
- A and B are subformulas of $A \wedge B$, $A \vee B$ $A \supset B$, e $A \equiv B$
- A is a subformula of $\neg A$
- if A is a subformula of B and B is a subformula of C, then A is a subformula of C.
- A is a proper subformula of B if A is a subformula of B and A is different from B.

Remark

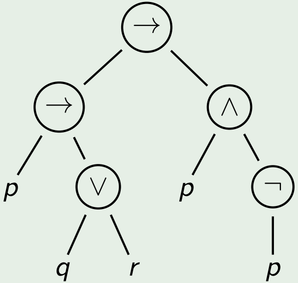
The subformulas of a formula represented as a tree correspond to all the different subtrees of the tree associated to the formula, one for each node.

Subformulas

Example

The subformulas of $(p \to (q \vee r)) \to (p \wedge \neg p)$ are

$$
\begin{array}{l}
(p \to (q \vee r)) \to (p \wedge \neg p) \\
(p \to (q \vee r)) \\
p \wedge \neg p \\
p \\
\neg p \\
q \vee r \\
q \\
r
\end{array}
$$



Proposition

Every formula has a finite number of subformulas

Interpretation of Propositional Logic

Definition (Interpretation)

A Propositional interpretation is a function $\mathcal{I} : \mathcal{P} \to \{\text{True}, \text{False}\}$

Remark

If $|\mathcal{P}|$ is the cardinality of $\mathcal{P}$, then there are $2^{|\mathcal{P}|}$ different interpretations, i.e. all the different subsets of $\mathcal{P}$. If $|\mathcal{P}|$ is finite then there is a finite number of interpretations.

Remark

A propositional interpretation can be thought as a subset S of $\mathcal{P}$, and $\mathcal{I}$ is the characteristic function of S, i.e., $A \in S$ iff $\mathcal{I}(A) = \text{True}$.

Interpretation of Propositional Logic

Example

	p	q	r	Set theoretic representation
$\mathcal{I}_1$	True	True	True	$\{p, q, r\}$
$\mathcal{I}_2$	True	True	False	$\{p, q\}$
$\mathcal{I}_3$	True	False	True	$\{p, r\}$
$\mathcal{I}_4$	True	False	False	$\{p\}$
$\mathcal{I}_5$	False	True	True	$\{q, r\}$
$\mathcal{I}_6$	False	True	False	$\{q\}$
$\mathcal{I}_7$	False	False	True	$\{r\}$
$\mathcal{I}_8$	False	False	False	$\{\}$

Satisfiability of a propositional formula

Definition ($\mathcal{I}$ satisfies a formula, $\mathcal{I} \models A$)

A formula A is true in/satisfied by an interpretation $\mathcal{I}$, in symbols $\mathcal{I} \models A$, according to the following inductive definition:

- If $P \in \mathcal{P}$, $\mathcal{I} \models P$ if $\mathcal{I}(P) = \text{True}$.
- $\mathcal{I} \models \neg A$ if not $\mathcal{I} \models A$ (also written $\mathcal{I} \not\models A$)
- $\mathcal{I} \models A \wedge B$ if, $\mathcal{I} \models A$ and $\mathcal{I} \models B$
- $\mathcal{I} \models A \vee B$ if, $\mathcal{I} \models A$ or $\mathcal{I} \models B$
- $\mathcal{I} \models A \rightarrow B$ if, when $\mathcal{I} \models A$ then $\mathcal{I} \models B$
- $\mathcal{I} \models A \equiv B$ if, $\mathcal{I} \models A$ iff $\mathcal{I} \models B$

Satisfiability of a propositional formula

Example (interpretation)

Let $\mathcal{P} = \{P, Q\}$.
$\mathcal{I}(P) = \mathit{True}$ and $\mathcal{I}(Q) = \mathit{False}$ can be also expressed with $\mathcal{I} = \{P\}$.

Example (Satisfiability)

Let $\mathcal{I} = \{P\}$. Check if $\mathcal{I} \models (P \wedge Q) \vee (R \rightarrow S)$:
Replace each occurrence of each primitive propositions of the formula with the truth value assigned by $\mathcal{I}$, and apply the definition for connectives.

$$(\text{True} \wedge \text{False}) \vee (\mathit{False} \rightarrow \text{False}) \tag{1}$$

$$\text{False} \vee \text{True} \tag{2}$$

$$\text{True} \tag{3}$$

Satisfiability of a propositional formula

Proposition

If for any propositional variable P appearing in a formula A, $\mathcal{I}(P) = \mathcal{I}'(P)$, then $\mathcal{I} \models A$ iff $\mathcal{I}' \models A$

Checking if $\mathcal{I} \models A$

Lazy evaluation algorithm (1/2)

$(A = p)$

check($\mathcal{I} \models p$)
  if $\mathcal{I}(p) = true$
    then return YES
  else return NO

$(A = B \wedge C)$

check($\mathcal{I} \models B \wedge C$)
  if check($\mathcal{I} \models B$)
    then return check($\mathcal{I} \models C$)
  else return NO

$(A = B \vee C)$

check($\mathcal{I} \models B \vee C$)
  if check($\mathcal{I} \models B$)
    then return YES
  else return check($\mathcal{I} \models C$)

Checking if $\mathcal{I} \models A$

Lazy evaluation algorithm (2/2)

$(A = B \rightarrow C)$

check($\mathcal{I} \models B \rightarrow C$)
  if check($\mathcal{I} \models B$)
    then return check($\mathcal{I} \models C$)
  else return YES

$(A = B \equiv C)$

check($\mathcal{I} \models B \equiv C$)
  if check($\mathcal{I} \models B$)
    then return check($\mathcal{I} \models C$)
  else return not(check($\mathcal{I} \models C$)

Formalizing English Sentences

Exercise

Let's consider a propositional language where p means *"Paola is happy"*, q means *"Paola paints a picture"*, and r means *"Renzo is happy"*.
Formalize the following sentences:

1. *"if Paola is happy and paints a picture then Renzo isn't happy"*
   $p \wedge q \rightarrow \neg r$
2. *"if Paola is happy, then she paints a picture"*
   $p \rightarrow q$
3. *"Paola is happy only if she paints a picture"*
   $\neg(p \wedge \neg q)$ which is equivalent to $p \rightarrow q$!!!

The precision of formal languages avoid the ambiguities of natural languages.

Valid, Satisfiable, and Unsatisfiable formulas

Definition

A formula A is

Valid if for all interpretations $\mathcal{I}$, $\mathcal{I} \models A$

Satisfiable if there is an interpretations $\mathcal{I}$ s.t., $\mathcal{I} \models A$

Unsatisfiable if for no interpretations $\mathcal{I}$, $\mathcal{I} \models A$

Proposition

A Valid $\longrightarrow$ *A satisfiable* $\longleftrightarrow$ *A not unsatisfiable*

A unsatisfiable $\longleftrightarrow$ *A not satisfiable* $\longrightarrow$ *A not Valid*

Valid, Satisfiable, and Unsatisfiable formulas

Proposition

if A is	*then* $\neg A$ *is*
Valid	*Unsatisfiable*
Satisfiable	*not Valid*
not Valid	*Satisfiable*
Unsatisfiable	*Valid*

Chesking Validity and (un)satisfiability of a formula

Truth Table

Checking (un)satisfiability and validity of a formula A can be done by enumerating all the interpretations which are relevant for S, and for each interpretation $\mathcal{I}$ check if $\mathcal{I} \models A$.

Example (of truth table)

A	B	C	$A \rightarrow (B \vee \neg C)$
true	true	true	true
true	true	false	true
true	false	true	false
true	false	false	true
false	true	true	true
false	true	false	true
false	false	true	true
false	false	false	true

Valid, Satisfiable, and Unsatisfiable formulas

Example

Satisfiable	$A \to A$	Valid
	$A \vee \neg A$	
	$\neg\neg A \equiv A$	
	$\neg(A \wedge \neg A)$	
	$A \wedge B \to A$	
	$A \to A \vee B$	
	$A \vee B$	Non Valid
	$A \to B$	
	$\neg(A \vee B) \to C$	
Uatisfiable	$A \wedge \neg A$	
	$\neg(A \to A)$	
	$A \equiv \neg A$	
	$\neg(A \equiv A)$	

Prove that the blue formulas are valid, that the magenta formulas are satisfiable but not valid, and that the red formulas are unsatisfiable.

Valid, Satisfiable, and Unsatisfiable sets of formulas

Definition

A set of formulas Γ is

- **Valid** if for all interpretations $\mathcal{I}$, $\mathcal{I} \models A$ for all formulas $A \in \Gamma$
- **Satisfiable** if there is an interpretations $\mathcal{I}$, $\mathcal{I} \models A$ for all $A \in \Gamma$
- **Unsatisfiable** if for no interpretations $\mathcal{I}$,, s.t. $\mathcal{I} \models A$ for all $A \in \Gamma$

Proposition

For any finite set of formulas Γ, (i.e., $\Gamma = \{A_1, \dots, A_n\}$ for some $n \geq 1$), Γ is valid (resp. satisfiable and unsatisfiable) if and only if $A_1 \wedge \cdots \wedge A_n$ is valid (resp, satisfiable and unsatisfiable).

Truth Tables: Example

Compute the truth table of $(F \vee G) \wedge \neg(F \wedge G)$.

F	G	$F \vee G$	$F \wedge G$	$\neg(F \wedge G)$	$(F \vee G) \wedge \neg(F \wedge G)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Intuitively, what does this formula represent?

Truth Tables

Recall some definitions

- Let F be a formula:
  - F is **valid** if every interpretation satisfies F
  - F is **satisfiable** if F is satisfied by some interpretation
  - F is **unsatisfiable** if there isn't any interpretation satisfying F

Truth Tables: Example (2)

Use the truth tables method to determine whether $(p \to q) \lor (p \to \neg q)$ is valid.

p	q	$p \to q$	$\neg q$	$p \to \neg q$	$(p \to q) \lor (p \to \neg q)$
T	T	T	F	F	**T**
T	F	F	T	T	**T**
F	T	T	F	T	**T**
F	F	T	T	T	**T**

The formula is valid since it is satisfied by every interpretation.

Truth Tables: Example (3)

Use the truth tables method to determine whether
$(\neg p \vee q) \wedge (q \rightarrow \neg r \wedge \neg p) \wedge (p \vee r)$ (denoted with F) is satisfiable.

p	q	r	$\neg p \vee q$	$\neg r \wedge \neg p$	$q \rightarrow \neg r \wedge \neg p$	$(p \vee r)$	F
T	T	T	T	F	F	T	F
T	T	F	T	F	F	T	F
T	F	T	F	F	T	T	F
T	F	F	F	F	T	T	F
F	T	T	T	F	F	T	F
F	T	F	T	T	T	F	F
F	F	T	T	F	T	T	**T**
F	F	F	T	T	T	F	F

There exists an interpretation satisfying F, thus F is satisfiable.

Truth Tables: Exercises

Compute the truth tables for the following propositional formulas:

- $(p \rightarrow p) \rightarrow p$
- $p \rightarrow (p \rightarrow p)$
- $p \vee q \rightarrow p \wedge q$
- $p \vee (q \wedge r) \rightarrow (p \wedge r) \vee q$
- $p \rightarrow (q \rightarrow p)$
- $(p \wedge \neg q) \vee \neg(p \leftrightarrow q)$

Truth Tables: Exercises

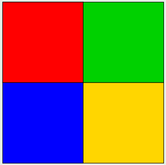
Use the truth table method to verify whether the following formulas are valid, satisfiable or unsatisfiable:

- $(p \rightarrow q) \wedge \neg q \rightarrow \neg p$
- $(p \rightarrow q) \rightarrow (p \rightarrow \neg q)$
- $(p \vee q \rightarrow r) \vee p \vee q$
- $(p \vee q) \wedge (p \rightarrow r \wedge q) \wedge (q \rightarrow \neg r \wedge p)$
- $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$
- $(p \vee q) \wedge (\neg q \wedge \neg p)$
- $(\neg p \rightarrow q) \vee ((p \wedge \neg r) \leftrightarrow q)$
- $(p \rightarrow q) \wedge (p \rightarrow \neg q)$
- $(p \rightarrow (q \vee r)) \vee (r \rightarrow \neg p)$

Formalization in Propositional Logic

Example (The colored blanket)

- $\mathcal{P} = \{B, R, Y, G\}$
- the intuitive interpretation of B (R, Y, and G) is that the blanket is completely blue (red, yellow and green)



Exercise

Find all the interpretations that, according to the intuitive interpretation given above, represent a possible situation. Consider the three cases in which

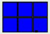
1. the blanket is composed of exactly 4 pieces, and yellow, red, blue and green are the only allowed colors;
2. the blanket can be composed of any number of pieces (at least 1), and yellow, red, blue and green are the only allowed colors;
3. the blanket can be composed of any number of pieces and there can be other colors.

Exercise (Solution)

1.
   - $\mathcal{I}_1 = \{B\}$ corrisponding to ;
   - $\mathcal{I}_2 = \{Y\}$ corrisponding to ;
   - $\mathcal{I}_3 = \{R\}$ corrisponding to ;
   - $\mathcal{I}_4 = \{G\}$ corrisponding to ;
   - $\mathcal{I}_5 = \emptyset$ corrisponding to any blanket that is not monochrome, e.g. ,  ...
   - $\mathcal{I}_6 = \{R, B\}$ does not correspond to any blanket, since a blanket cannot be both completely blue and red. More in general all the interpretations that satisfies more than one proposition do not correspond to any real situation.
   - ...

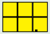
Formalization in Propositional Logic

Exercise (Solution)

2. - $\mathcal{I}_1 = \{B\}$ corrisponding to any blue blankets, no matter its shape, e.g. , , and .
   - $\mathcal{I}_2 = \{Y\}$ corrisponding to any blue blankets, no matter its shape, e.g. , , and .
   - ...
   - $\mathcal{I}_5 = \emptyset$ corresponds to any blanket which is not monochrome no matter of its shape, e.g., , , and .
   - $\mathcal{I}_6 = \{R, B\}$ does not correspond to any blanket, since a blanket cannot be both completely blue and red. More in general all the interpretations that satisfies more than one proposition do not correspond to any real situation.
   - ...

Formalization in Propositional Logic

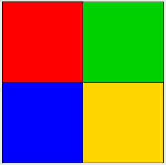
Exercise (Solution)

3.
   - $\mathcal{I}_1 = \{B\}$ corrisponding to any blue blankets, no matter its shape, n e.g. , , and .
   - $\mathcal{I}_2 = \{Y\}$ corrisponding to any yellow blankets, no matter its shape, e.g. , , and .
   - ...
   - $\mathcal{I}_5 = \emptyset$ corresponds to any blanket which is neither completely blue, red, yellow, nor green, no matter of its shape, e.g.,   , and .
   - $\mathcal{I}_6 = \{R, B\}$ does not correspond to any blanket, since a blanket cannot be both completely blue and red. More in general all the interpretations that satisfies more than one proposition do not correspond to any real situation.
   - ...

Formalization in Propositional Logic

Example (The colored blanket)

- $\mathcal{P} = \{B, R, Y, G\}$
- the intuitive interpretation of B (R, Y, and G) is that at least one piece of the blanket is blue (red, yellow and green)



Exercise

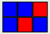
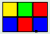
Find all the interpretations that, according to the intuitive interpretation given above, represent a realistic situation. Consider the three cases in which:

1. the blanket is composed of exactly 4 pieces, and yellow, red, blue and green are the only allowed colors;
2. the blanket can be composed of any number of pieces (at least one), and yellow, red, blue, and green are the only allowed colors;
3. the blanket can be composed of any number of pieces and there can be other colors.

Exercise (Solution)

1.
   - $\mathcal{I}_1 = \{B\}$ corresponding to the blue blanket 
   - $\mathcal{I}_2 = \{Y\}$ corresponding to the yellow blanket ,
   - ...
   - $\mathcal{I}_5 = \emptyset$ corresponds to no (empty) blanket
   - $\mathcal{I}_6 = \{R, B\}$ corresponding to the red and blue blanket no matter of the color position , e.g., ,  and 
   - ...
   - $\mathcal{I}_{16} = \{R, B, Y, G\}$ corresponding to the blankets containing all the colors, no matter of the color position, e.g., , , and .

Exercise (Solution)

2.
   - $\mathcal{I}_1 = \{B\}$ corresponding to any blue blanket, no matter of the shape, e.g., , .
   - $\mathcal{I}_2 = \{Y\}$ corresponding to any yellow blanket, no matter of the shape, e.g., , .
   - ...
   - $\mathcal{I}_5 = \emptyset$ corresponds to none blanket
   - $\mathcal{I}_6 = \{R, B\}$ corresponding to the red and blue blankets no matter of the color position and the shape (provided that they contain at least two pieces) e.g.,    and 
   - ...
   - $\mathcal{I}_{16} = \{R, B, Y, G\}$ corresponding to the blankets containing all the colors, no matter of the color position (provided that they contain at least 4 pieces), e.g., , , and .

Logical consequence

Definition (Logical consequence)

A formula A is a logical consequence of a set of formulas Γ, in symbols

$$\Gamma \models A$$

Iff for any interpretation $\mathcal{I}$ that satisfies all the formulas in Γ, $\mathcal{I}$ satisfies A,

Example (Logical consequence)

- $p \models p \vee q$
- $q \vee p \models p \vee q$
- $p \vee q, p \rightarrow r, q \rightarrow r \models r$
- $p \rightarrow q, p \models q$
- $p, \neg p \models q$

Proving Logical consequence in a direct manner

Example

Proof of $p \models p \vee q$ Suppose that $\mathcal{I} \models p$, then by definition $\mathcal{I} \models p \vee q$.

Proof of $q \vee p \models p \vee q$ Suppose that $\mathcal{I} \models q \vee p$, then either $\mathcal{I} \models q$ or $\mathcal{I} \models p$. In both cases we have that $\mathcal{I} \models p \vee q$.

Proof of $p \vee q, p \rightarrow r, q \rightarrow r \models r$ Suppose that $\mathcal{I} \models p \vee q$ and $\mathcal{I} \models p \rightarrow r$ and $\mathcal{I} \models q \rightarrow r$. Then either $\mathcal{I} \models p$ or $\mathcal{I} \models q$. In the first case, since $\mathcal{I} \models p \rightarrow r$, then $\mathcal{I} \models r$, In the second case, since $\mathcal{I} \models q \rightarrow r$, then $\mathcal{I} \models r$.

Proof of $p, \neg p \models q$ Suppose that $\mathcal{I} \models \neg p$, then not $\mathcal{I} \models p$, which implies that there is no $\mathcal{I}$ such that $\mathcal{I} \models p$ and $\mathcal{I} \models \neg p$. This implies that all the interpretations that satisfy p and $\neg p$ (actually none) satisfy also q.

Proof of $(p \wedge q) \vee (\neg p \wedge \neg q) \models p \equiv q)$ Left as an exercise

Proof of $(p \rightarrow q) \models \neg p \vee q$ Left as an exercise

Proving Logical consequence using the truth tables

Use the truth tables method to determine whether $p \wedge \neg q \rightarrow p \wedge q$ is a logical consequence of $\neg p$.

p	q	$\neg p$	$p \wedge \neg q$	$p \wedge q$	$p \wedge \neg q \rightarrow p \wedge q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	**T**	F	F	**T**
F	F	**T**	F	F	**T**

Truth Tables: Exercises

Use the truth table method to verify whether the following logical consequences and equivalences are correct:

- $(p \rightarrow q) \models \neg p \rightarrow \neg q$
- $(p \rightarrow q) \wedge \neg q \models \neg p$
- $p \rightarrow q \wedge r \models (p \rightarrow q) \rightarrow r$
- $p \vee (\neg q \wedge r) \models q \vee \neg r \rightarrow p$
- $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- $(p \vee q) \wedge (\neg p \rightarrow \neg q) \equiv q$
- $(p \wedge q) \vee r \equiv (p \rightarrow \neg q) \rightarrow r$
- $(p \vee q) \wedge (\neg p \rightarrow \neg q) \equiv p$
- $((p \rightarrow q) \rightarrow q) \rightarrow q \equiv p \rightarrow q$

Logical Equivalence

Definition

Logical Equivalence Two formulas F and G are **logically equivalent** (denoted with $F \equiv G$) if for each interpretation $\mathcal{I}$, $\mathcal{I}(F) = \mathcal{I}(G)$.

Truth Tables: Example (5)

Use the truth tables method to determine whether $p \rightarrow (q \wedge \neg q)$ and $\neg p$ are logically equivalent.

p	q	$q \wedge \neg q$	$p \rightarrow (q \wedge \neg q)$	$\neg p$
T	T	F	**F**	**F**
T	F	F	**F**	**F**
F	T	F	**T**	**T**
F	F	F	**T**	**T**

Properties of propositional logical consequence

Proposition

If Γ and Σ are two sets of propositional formulas and A and B two formulas, then the following properties hold:

Reflexivity $\{A\} \models A$

Monotonicity *If* $\Gamma \models A$ *then* $\Gamma \cup \Sigma \models A$

Cut *If* $\Gamma \models A$ *and* $\Sigma \cup \{A\} \models B$ *then* $\Gamma \cup \Sigma \models B$

Compactness *If* $\Gamma \models A$*, then there is a finite subset* $\Gamma_0 \subseteq \Gamma$*, such that* $\Gamma_0 \models A$

Deduction theorem *If* $\Gamma, A \models B$ *then* $\Gamma \models A \rightarrow B$

Refutation principle $\Gamma \models A$ *iff* $\Gamma \cup \{\neg A\}$ *is unsatisfiable*

Reflexivity $\{A\} \models A$.

PROOF: For all $\mathcal{I}$ if $\mathcal{I} \models A$, then $\mathcal{I} \models A$.

Monotonicity If $\Gamma \models A$ then $\Gamma \cup \Sigma \models A$

PROOF: For all $\mathcal{I}$ if $\mathcal{I} \models \Gamma \cup \Sigma$, then $\mathcal{I} \models \Gamma$, by hypothesis ($\Gamma \models A$) we can infer that $\mathcal{I} \models A$, and therefore that $\Gamma \cup \Sigma \models A$

Cut If $\Gamma \models A$ and $\Sigma \cup \{A\} \models B$ then $\Gamma \cup \Sigma \models B$.

PROOF: For all $\mathcal{I}$, if $\mathcal{I} \models \Gamma \cup \Sigma$, then $\mathcal{I} \models \Gamma$ and $\mathcal{I} \models \Sigma$. The hypothesis $\Gamma \models A$ implies that $\mathcal{I} \models A$. Since $\mathcal{I} \models \Sigma$, then $\mathcal{I} \models \Sigma \cup \{A\}$. The hypothesis $\Sigma \cup \{A\} \models B$, implies that $\mathcal{I} \models B$. We can therefore conclude that $\Gamma \cup \Sigma \models B$.

Compactness If $\Gamma \models A$, then there is a finite subset $\Gamma_0 \subseteq \Gamma$, such that $\Gamma_0 \models A$.

PROOF: Let $\mathcal{P}_A$ be the primitive propositions occurring in A. Let $\mathcal{I}_1, \dots, \mathcal{I}_n$ (with $n \leq 2^{|\mathcal{P}_A|}$), be all the interpretations of the language $\mathcal{P}_A$ that do not satisfy A. Since $\Gamma \models A$, then there should be $\mathcal{I}'_1, \dots, \mathcal{I}'_n$ interpretations of the language of Γ, which are extensions of $\mathcal{I}_1, \dots, \mathcal{I}_n$, and such that $\mathcal{I}'_k \not\models \gamma_k$ for some $\gamma_k \in \Gamma$. Let $\Gamma_0 = \{\gamma_1, \dots, \gamma_k\}$. Then $\Gamma_0 \models A$. Indeed if $\mathcal{I} \models \Gamma_0$ then $\mathcal{I}$ is an extension of an interpretation J of $\mathcal{P}_A$ that satisfies A, and therefore $\mathcal{I} \models A$.

Deduction theorem If $\Gamma, A \models B$ then $\Gamma \models A \rightarrow B$

PROOF: Suppose that $\mathcal{I} \models \Gamma$. If $\mathcal{I} \not\models A$, then $\mathcal{I} \models A \rightarrow B$. If instead $\mathcal{I} \models A$, then by the hypothesis $\Gamma, A \models B$, implies that $\mathcal{I} \models B$, which implies that $\mathcal{I} \models B$. We can therefore conclude that $\mathcal{I} \models A \rightarrow B$.

Refutation principle $\Gamma \models A$ iff $\Gamma \cup \{\neg A\}$ is unsatisfiable

PROOF:

($\Longrightarrow$) Suppose by contradiction that $\Gamma \cup \{\neg A\}$ is satisfiable. This implies that there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \Gamma$ and $\mathcal{I} \models \neg A$, i.e., $\mathcal{I} \not\models A$. This contradicts that fact that for all interpretations that satisfies Γ, they satisfy A

($\Longleftarrow$) Let $\mathcal{I} \models \Gamma$, then by the fact that $\Gamma \cup \{\neg A\}$ is unsatisfiable, we have that $\mathcal{I} \not\models \neg A$, and therefore $\mathcal{I} \models A$. We can conclude that $\Gamma \models A$.

Propositional theory

Definition (Propositional theory)

A theory is a set of formulas closed under the logical consequence relation. I.e. T is a theory iff $T \models A$ implies that $A \in T$

Example (Of theory)

- T_1 is the set of valid formulas $\{A | A \text{ is valid}\}$
- T_2 is the set of formulas which are true in the interpretation $\mathcal{I} = \{P, Q, R\}$
- T_3 is the set of formulas which are true in the set of interpretations $\{I_1, I_2, I_3\}$
- T_4 is the set of all formulas

Show that T_1, T_2, T_3 and T_4 are theories

Propositional theory (2)

Example (Of non theory)

- N_1 is the set $\{A, A \rightarrow B, C\}$
- N_2 is the set $\{A, A \rightarrow B, B, C\}$
- N_3 is the set of all formulas containing P

Show that N_1, N_2 and N_3 are not theories

Axiomatization

Remark

A propositional theory always contains an infinite set of formulas. Indeed any theory T contains at least all the valid formulas. which are infinite) (e.g.,$A \rightarrow A$ for all formulas A)

Definition (Set of axioms for a theory)

A set of formulas Ω is a set of axioms for a theory T if for all $A \in T$, $\Omega \models A$.

Definition

Finitely axiomatizable theory A theory T is finitely axiomatizable if it has a finite set of axioms.

Propositional theory (cont'd)

Definition (Logical closure)

For any set Γ, $cl(\Gamma) = \{A | \Gamma \models A\}$

Proposition (Logical closure)

For any set Γ, the logical closure of Γ, $cl(\Gamma)$ is a theory

Proposition

Γ is a set of axioms for $cl(\Gamma)$.

Axioms and theory - intuition

Compact representation of knowldge

The axiomatization of a theory is a compact way to represent a set of interpretations, and thus to represent a set of possible (acceptable) world states. In other words is a way to represent all the knowledge we have of the real world.

minimality

The axioms of a theory constitute the basic knowledge, and all the *generable knolwledge* is obtained by logical consequence. An important feature of a set of axioms, is that they are minimal, i.e., no axioms can be derived from the others.

Axioms and theory - intuition

Example

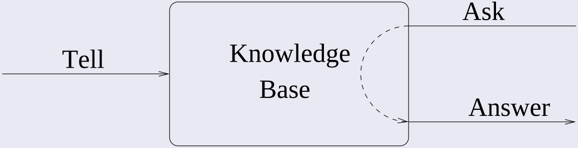
$$
\begin{array}{l}
\mathit{Pam\_Attends\_Logic\_Course} \\
\mathit{John\_is\_a\_Phd\_Student} \\
\mathit{Pam\_Attends\_Logic\_Course} \rightarrow \mathit{Pam\_is\_a\_Ms\_Student} \vee \mathit{Pam\_is\_a\_PhD\_Student} \\
\mathit{Pam\_is\_a\_Ms\_Student} \rightarrow \neg \mathit{Pam\_is\_a\_Ba\_Student} \\
\mathit{Pam\_is\_a\_PhD\_Student} \rightarrow \neg \mathit{Pam\_is\_a\_Ba\_Student} \\
\neg(\mathit{John\_is\_a\_Phd\_Student} \wedge \mathit{John\_is\_a\_Ba\_Student})
\end{array}
$$

The axioms above constitute the basic knowledge about the people that attend logic course. The facts $\neg \mathit{Pam\_is\_a\_Bs\_Student}$ and $\neg \mathit{John\_is\_a\_Bs\_Student}$ don't need to be added to this basic knowledge, as they can be derived via logical consequence.

Logic based systems

A logic-based system for representing and reasoning about knowledge is composed by a Knowledge base and a Reasoning system. A knowledge base consists of a finite collection of formulas in a logical language. The main task of the knowledge base is to answer queries which are submitted to it by means of a Reasoning system

Logic based system for knowledge representation



Tell: this action incorporates the new knowledge encoded in an axiom (formula). This allows to build a KB.

Ask: allows to query what is known, i.e., whether a formula ϕ is a logical consequences of the axioms contained in the KB ($KB \models \phi$)

Propositional theory (cont'd)

Proposition

Given a set of interpretations S, the set of formulas A which are satisfied by all the interpretations in S is a theory. i.e.

$$T_S = \{A | \mathcal{I} \models A \text{ for all } \mathcal{I} \in S\}$$

is a theory.

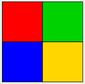
Knowledge representation problem

Given a set of interpretations S which correspond to admissible situations find a set of axioms Ω for T_S.

Propositional theories examples

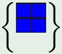
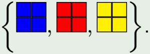
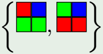
Example (The colored blanket)

- $\mathcal{P} = \{B, R, Y, G\}$
- the intuitive interpretation of B (R, Y, G) is that the blanket contains at least blue (red, yellow, green) piece.



Exercise

Provide an axiomatization for the following set of blankets. Hypothesis: (i) blankets are 2x2; (ii) yellow, red, blue, and green are the only colours.

1. 
2. 
3. 
4. 
4. the set of blankets that never combine blue with red, or green with yellow
5. the set of blankets that contain at least three colors
6. the set of blankets that contain at most two colors
7. the set of blankets that contain some blue pieces whenever a green pieces is present