Mathematical Logic

11. Modal Logics - relation with FOL

Luciano Serafini

FBK-IRST, Trento, Italy

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Kripke models and First order structures

- A Kripke model $\mathcal{I}$ (as defined in the previous slides) is equal to the pair (F, V) where F is a frame (W, R) and V is a truth assignment $V : \mathcal{P} \rightarrow 2^W$.
- A Kripke model can be seen as a first order interpretation $I_{FOL} = (\Delta^{I_{FOL}}, (,)^{I_{FOL}})$ of the following language:
  - a unary predicate $P(x)$ for every proposition $P \in \mathcal{P}$ Indeed V associated to each $P \in \mathcal{P}$ a set of worlds;
  - the binary relation $r(x, y)$ for the accessibility relation, which is a binary relation on the set of worlds.

  Intuitively, $P(x)$ represents the facts that P is true in the world x and $r(x, y)$ represents the fact that the world y is accessible form the world x.
- $\Delta^{I_{FOL}} = W$, i.e., the domain of interpretation is the set of possible worlds. $r^{I_{FOL}}$ is the accessibility relation R, and $P^{\mathcal{I}}$ is equal to $V(P)$.

Modal formulas and First order formulas

- $I, w \models P$ means that I satisfies the atomic formula P in the world w. In the corresponding first order language, this can be expressed by the fact that $I_{FOL} \models P(x)[x := w]$
- $I, w \models P \wedge Q$ means that I satisfies the $P \wedge Q$ in the world w. In the corresponding first order language, this can be expressed by the fact that $I_{FOL} \models P(x) \wedge Q(x)[x := w]$
- $I, w \models \Box P$ means that I satisfies P in all the worlds w' accessible from w. In the corresponding first order language, this can be expressed by the fact that $I_{FOL} \models \forall y(r(x, y) \supset P(y))[x := w]$
- $I, w \models \Diamond P$ means that I satisfies P in at least one world w' accessible from w. In the corresponding first order language, this can be expressed by the fact that $I_{FOL} \models \exists y(r(x, y) \wedge P(y))[x := w]$
- $I, w \models \Diamond\Box P$ means that there is a world w' accessible from w such that for all worlds w'' accessible from w' w'' satisfies P. In FOL this can be expressed by the following formula $I_{FOL} \models \exists y(r(x, y) \wedge \forall z(r(y, z) \supset P(z)))$

Standard translation of Modal formulas into First order formulas

$$
\begin{aligned}
\mathrm{ST}^x(P) &= P(x) \\
\mathrm{ST}^x(A \circ B) &= \mathrm{ST}^x(A) \circ \mathrm{ST}^x(B) \;\; \text{with } \circ \in \{\wedge, \vee, \supset, \equiv\} \\
\mathrm{ST}^x(\neg A) &= \neg \mathrm{ST}^x(A) \\
\mathrm{ST}^x(\Box A) &= \forall y(R(x,y) \supset \mathrm{ST}^y(A)) \\
\mathrm{ST}^x(\Diamond A) &= \exists y(R(x,y) \wedge \mathrm{ST}^y(A))
\end{aligned}
$$

Example

$\mathrm{ST}^x(\Box\Box P \wedge \Box\Diamond Q \supset \Box\Diamond(P \wedge Q))$ is equal to

$$
\begin{array}{rl}
\forall y(r(x,y) \supset (\forall z(r(y,z) \supset P(z)))) \wedge & \mathrm{ST}^x(\Box\Box P) \\
\forall y(r(x,y) \supset (\exists z(r(y,z) \wedge Q(z)))) \;\supset & \mathrm{ST}^x(\Box\Diamond Q) \\
\forall y(r(x,y) \supset (\exists z(r(y,z) \wedge P(z) \wedge Q(z)))) & \mathrm{ST}^x(\Box\Diamond(P \wedge Q))
\end{array}
$$

The standard translation

Theorem

If $I = ((W, R), V)$ is a Kripke model, I_{FOL} the corresponding first order interpretation of the translated language, then, for every modal formula ϕ

$$I \models \phi \text{ if and only if } I_{FOL} \models \forall x ST^x(\phi)$$

Proof.

The proof is by induction on the complexity of ϕ.

Base case Suppose that ϕ is the atomic formula P.

$$\begin{aligned} I \models P \quad & \textit{iff} \quad \text{for all } w \in W,\ I, w \models P \\ & \textit{iff} \quad V(P) = W \\ & \textit{iff} \quad I_{FOL}(P) = \Delta^{I_{FOL}} \\ & \textit{iff} \quad I_{FOL} \models \forall x P(x) \end{aligned}$$

□

Relation between the expressivity of Logics

Propositional Logic (Prop): Propositional variables $p_1, p_2, \ldots,$ and propositional connectives $\wedge, \vee, \supset, \equiv,$ and $\neg$

Modal Logic (Mod) $=$ Prop $+$ modal operators $\Box$ and $\Diamond$

First-order logic (Fol) $=$ Prop $+$ constants, function, and relations, and quantifiers $\forall$ and $\exists$

The following relations between the expressivity of the three logic above hold:

$$Prop \subset Mod \subset Fol$$

- every propositional formula is a formula of modal logic, but not viceversa. For instance $\Box P$ does not have any correspondence in propositional logic.
- every modal formula can be translated under the standard translation into a first order formula with at most 2 variables. On the other hand there are first order formulas that cannot be translated back into modal formulas, for instance $\forall xyz\ P(x, y, f(z))$ or $\forall xy(P(x, y) \vee P(y, x))$.