Mathematical Logic

20. Normal Modal Logics - K, T, 4, 5, and more

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28 November 2013

Exercise

Prove that the following formulas are valid. I.e., they are valid in avery frame.

1. $\Diamond(\phi \vee \psi) \supset (\Diamond\phi \vee \Diamond\psi)$
2. $\Diamond(\phi \wedge \psi) \supset \Diamond\phi$
3. $\Box\phi \wedge \Diamond\psi \supset \Diamond(\phi \wedge \psi)$
4. $\Diamond\phi \supset (\Box\psi \supset \neg\Box\neg\psi)$
5. $\Diamond^n\phi \supset \neg\Box^m\bot$ for $m \leq n$
6. $\Diamond\phi \vee \Diamond\neg\phi \vee \Box\bot$
7. $\Box\phi \vee \Box\neg\phi \vee (\Diamond\phi \wedge \Diamond\neg\phi)$

Solution

1. $\Diamond(\phi \vee \psi) \supset (\Diamond\phi \vee \Diamond\psi)$

    - *If* $\mathcal{M}, w \models \Diamond(\phi \vee \psi)$,
    - *then there is a* w' *accessible from* w *(i.e.,* wRw'*) such that* $\mathcal{M}, w' \models \phi \vee \psi$
    - *which implies that either* $\mathcal{M}, w' \models \phi$ *or* $\mathcal{M}, w' \models \psi$
        - *If* $\mathcal{M}, w' \models \phi$, *then, since* wRw' *we have that* $\mathcal{M}, w \models \Diamond\phi$
        - *If* $\mathcal{M}, w' \models \psi$, *then, since* wRw' *we have that* $\mathcal{M}, w \models \Diamond\psi$
    - *in both cases we have that* $\mathcal{M}, w \models \Diamond\phi \vee \Diamond\psi$.
    - *therefore we conclude that* $\mathcal{M}, w \models \Diamond(\phi \vee \psi) \supset (\Diamond\phi \vee \Diamond\psi)$.

2. $\Diamond(\phi \wedge \psi) \supset \Diamond\phi$

    - *If* $\mathcal{M}, w \models \Diamond(\phi \wedge \psi)$,
    - *then there is a* w' *with* wRw' *such that* $\mathcal{M}, w' \models \phi \wedge \psi$
    - *which implies that* $\mathcal{M}, w' \models \phi$
    - *since* wRw' *we have that* $\mathcal{M}, w \models \Diamond\phi$
    - *therefore we conclude that* $\mathcal{M}, w \models \Diamond(\phi \wedge \psi) \supset \Diamond\phi$

Solution

1. $\Box\phi \wedge \Diamond\psi \supset \Diamond(\phi \wedge \psi)$
   - *If* $\mathcal{M}, w \models \Box\phi \wedge \Diamond\psi$ *then* $\mathcal{M}.w \models \Box\phi$ *and* $\mathcal{M}, w \models \Diamond\psi$.
   - $\mathcal{M}, w \models \Diamond\psi$ *implies that there is a* w' *with* wRw' *such that* $\mathcal{M}, w' \models \psi$.
   - $\mathcal{M}, w \models \Box\psi$ *implies that for all world accessible from* w*, and therefore also for* w'*,* $\mathcal{M}, w' \models \phi$,
   - *this allows to conclude that* $\mathcal{M}, w' \models \phi \wedge \psi$
   - *and therefore, since* wRw'*,* $\mathcal{M}, w \models \Diamond(\phi \wedge \psi)$.

Solution

1. $\Diamond\phi \supset (\Box\psi \supset \neg\Box\neg\psi)$

   - *$\mathcal{M}, w \models \Diamond\phi$ implies that there is a world w' with wRw' such that $\mathcal{M}, w' \models \phi$. (In the proof we will use only the fact that there is a world w' accessible from w, the fact that $\mathcal{M}, w' \models \phi$ is completely irrelevant)*
   - *suppose that $\mathcal{M}, w \models \Box\psi$,*
   - *than since wRw', and for all world accessible from w, ψ must be true, we have that $\mathcal{M}, w' \models \psi$,*
   - *which implies that $\mathcal{M}, w' \not\models \neg\psi$*
   - *the fact that w has an accessible world w' with $\mathcal{M}, w' \not\models \neg\psi$ implies that $\mathcal{M}, w \not\models \Box\neg\psi$*
   - *which implies that $\mathcal{M}, w \models \neg\Box\neg\psi$*
   - *we can therefore conclude that $\mathcal{M}, w \models \Box\psi \supset \neg\Box\neg\psi$ under that assumption that $\mathcal{M}, w \models \Diamond\phi$*
   - *and therefore we conclude that $\mathcal{M}, w \models \Diamond\phi \supset (\Box\psi \supset \neg\Box\neg\psi)$.*

Solution

1. $\Diamond^n\phi \supset \neg\Box^m\bot$ for $m \leq n$

For $n, m \geq 0$, $\Diamond^n$ stands for $\overbrace{\Diamond\ldots\Diamond}^{n\ times}$.

- $\Diamond^1\top$ is equal to $\Diamond\top$, and we have that $\mathcal{M}, w \models \Diamond^1\top$, if there is a possible world w_1 accessible from w (i.e., wRw_1)
- $\Diamond^2\top$ is equal to $\Diamond\Diamond\top$. therefore $\mathcal{M}, w \models \Diamond^2\top$ if there is a world w_1 with wRw_1 such that $\mathcal{M}, w_1 \models \Diamond\top$, which in turn is true if there is a world w_2 with w_1Rw_2.
- continuing reasoning like above, we have that $\mathcal{M}, w \models \Diamond^n\top$ if there are n worlds $w_1, \ldots, w_n$ such that wRw_1,w_1Rw_2 $\ldots w_{n-1}Rw_n$, i.e., if there is a path (it can be also circular) of n steps.

Solution

1. $\Diamond^n \phi \supset \neg\Box^m\bot$ *for* $m \leq n$ *(cont'd)*

The formula $\Box^m$ *stands for* $\overbrace{\Diamond \dots \Box}^{m \text{ times}}$.

- $\mathcal{M}, w \models \not\Box^1\bot$ *means that* $\mathcal{M}, w \models \not\Box\bot$,
- *which implies that* $\mathcal{M}, w \not\models \Box\bot$. *Notice that the only case in which* $\mathcal{M}, \models \Box\bot$ *is when there is no world accessible from* w.
- *therefore,* $\mathcal{M}, w \not\models \Box\bot$ *implies that there iw a world* w_1 *accessible from* w*, i.e.,* wRw'.
- $\mathcal{M}, w \models \neg\Box^2\bot$ *means that* $\mathcal{M}, w \not\models \Box^2\bot$,
- *this implies that there is a* w_1 *accessible from* w *such that* $\mathcal{M}, w_1 \not\models \Box\bot$, *which in turns implies that there is a world* w_2 *accessible form* w_1.
- *iterating* m *times the above reasoning we have that* $\mathcal{M}, w \models \neg\Box^m\bot$ *if there are* m *worlds* $w_1, \dots, w_m$ *with* wRw_1, w_1Rw_2, $\dots$, $w_{m-1}Rw_m$. *i.e., if there is a path of length* m.

Solution

1. $\Diamond^n\phi \supset \neg\Box^m\bot$ *for* $m \leq n$ *(cont'd)*

   Summarizing:

   - $\mathcal{M}, w \models \Diamond^n\top$ *if there is a path of length* n
   - $\mathcal{M}, w \models \neg\Box^m\bot$ *if there is a path of length* m
   - *the fact that* $m \leq n$ *implies that if there is a path of length* n *there is also a path of length* m *(just take the first* m *steps of the path of length* n*)*
   - *which implies that* $\mathcal{M}, w \models \Diamond^n\top \supset \neg\Box^m\bot$ *with* $m \leq n$.

2. $\Diamond\phi \vee \Diamond\neg\phi \vee \Box\bot$

   - *Suppose that* $\mathcal{M}, w \not\models \Box\bot$ *then*
   - *there is a world* w', *with* wRw'.
   - *we have that either* $\mathcal{M}, w' \models \phi$ *or* $\mathcal{M}, w' \not\models \phi$
   - *in the first case we have that* $\mathcal{M}, w \models \diamond\phi$
   - *in the second case* $\mathcal{M}, w' \models \neg\phi$ *and therefore* $\mathcal{M}, w \models \Diamond\neg\phi$.
   - *This implies that either* $\Box\bot$ *or* $\Diamond\phi$ *or* $\Diamond\neg\phi$ *is true in* w.
   - *and therefore* $\mathcal{M}, w \models \Diamond\phi \vee \Diamond\neg\phi \vee \Box\bot$.

Solution

1. $\Box\phi \lor \Box\neg\phi \lor (\Diamond\phi \land \Diamond\neg\phi)$
   - *Suppose that* $\mathcal{M}, w \not\models \Box\phi$
   - *This implies that there is a world* w_1 *accessible from* w *such that* $\mathcal{M}, w_1 \not\models \phi$,
   - *this implies that* $\mathcal{M}, w_1 \models \neg\phi$ *and therefore* $\mathcal{M}, w \models \Diamond\neg\phi$
   - *Suppose that* $\mathcal{M}, w \not\models \Box\neg\phi$ *then there is a world* w_2 *accessible from* w *such that* $\mathcal{M}, w_2 \not\models \neg\phi$,
   - *this implies that* $\mathcal{M}, w_2 \models \phi$ *and therefore that* $\mathcal{M}, w \models \Diamond\phi$.
   - *we can conclude that if* $\mathcal{M}, w \not\models \Box\phi$ *and* $\mathcal{M}, w \not\models \Box\neg\phi$, *then* $\mathcal{M}, w \models \Diamond\phi \land \diamond\neg\phi$.
   - *which implies that* $\mathcal{M}, w \models \Box\phi \lor \Box\neg\phi \lor (\Diamond\phi \land \Diamond\neg\phi)$.

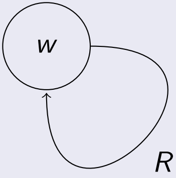
Exercise

Find a frame in which the following formulas are valid:

1. $p \equiv \Box p$
2. $p \equiv \Diamond p$
3. $\Diamond p \equiv \Box p$
4. $\Diamond(p \wedge \Diamond q) \supset q \wedge \Diamond p)$

Solution

The following frame is such that all the above formulas are valid in it.



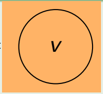
Exercises on Modal Logics

Exercise

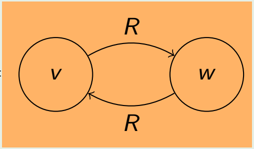
Check if the following formulas are valid in the frames below; If they are not valid find a truth assignment of the propositional variables in the worlds, and a world for which the formula is not satisfied.

1. $p \equiv \Box p$
2. $p \equiv \Diamond p$
3. $\Diamond p \equiv \Box p$
4. $\Diamond(p \wedge \Diamond q) \supset q \wedge \Diamond p$

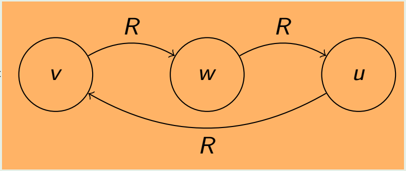
$\mathcal{F}_1 =$



$\mathcal{F}_2 =$



$\mathcal{F}_2 =$



Exercise

Show that the following formulas are non-valid by constructing a counterexample, i.e., a frame and an assignment to the propositional variable and a world that falsify them:

1. $\Box\bot$
2. $\Diamond p \supset \Box p$
3. $p \supset \Box\Diamond p$
4. $\Diamond\Box p \supset \Box\Diamond p$
5. $\Box p \supset p$

Material implication and strict implication

Paradoxes of implication in PL (material implication)

In PL we have that $\neg A \supset (A \supset B)$ and that $B \supset (A \supset B)$ are valid formulas. These facts are very counterintuitive. E.g., the following statements are valid according to the formalization in PL:

- if it's raining, then the fact that it is sunny implies that Mario owns a Ferrari
- if Mario owns a Ferrari then this is implied by the fact that it is sunny

Use modal logics to solve these paradoxes (strict implication)

C. I. Lewis in 1917 proposes a different formalization of implication,

- According to Lewis "A implies B" requires that it is impossible that both A and $\neg B$ are true.

In the moder notation of modal logic Lewis notion of implication can be formalized by:

$$\neg\Diamond(A \wedge \neg B)$$

Which is equivalent to

$$\Box(A \supset B)$$

Material implication and strict implication

Exercise

Suppose that we define $A \Rightarrow B$ as $\Box(A \supset B)$ show that the following formulas are not valid in the class of Kripke frames.

- $\neg A \Rightarrow (A \Rightarrow B)$
- $B \Rightarrow (A \Rightarrow B)$

Solution

This corresponds to show that the formulas

- $\Box(\neg A \supset \Box(A \supset B))$
- $\Box(B \supset \Box(A \supset B))$

are not valid in the class of Kripke Frames. I.e., that there is a Kripke model $\mathcal{M} = \langle \mathcal{F}, \mathcal{I} \rangle$ and a world w such that

- $\mathcal{M}, w \not\models \Box(\neg A \supset \Box(A \supset B))$
- $\mathcal{M}, w \not\models \Box(B \supset \Box(A \supset B))$

Material implication and strict implication

Solution

$$\begin{tikzcd} w \arrow[r, "R"] & w' \arrow[r, "R"] & w'' \end{tikzcd}$$

$$\begin{array}{ccc}
\neg\Box(\neg A \supset \Box(A \supset B)) & \neg A & A \\
 & \neg\Box(A \supset B) & \neg B \\
 & \neg(\neg A \supset \Box(A \supset B)) & \neg(A \supset B)
\end{array}$$

$$\begin{tikzcd} w \arrow[r, "R"] & w' \arrow[r, "R"] & w'' \end{tikzcd}$$

$$\begin{array}{ccc}
\neg\Box(B \supset \Box(A \supset B)) & B & A \\
 & \neg\Box(A \supset B) & \neg B \\
 & \neg(B \supset \Box(A \supset B)) & \neg(A \supset B)
\end{array}$$

Properties of accessibility relation

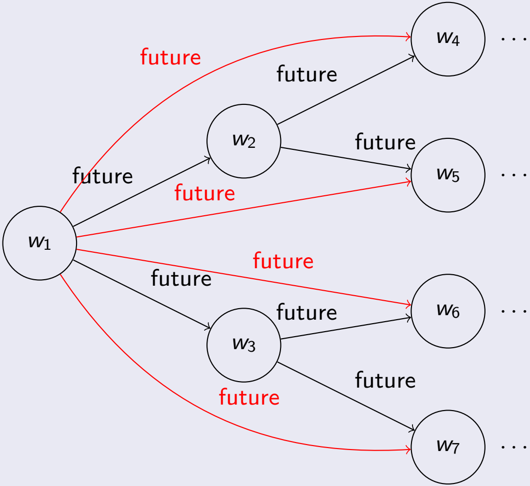
Depending on the intuitive interpretation of the accessibility relation between the possible worlds, we need to impose different properties on it. For instance:

Temporal logic

- In modal temporal logics states of the world are ordered according to a past-future relation. This order is encoded in the accessibility relation.
- wRw' means that if we are in the state of the worlds w then in the future we *could* reach the state w'.
- Note that, if w' is a future state of w, then every future state w'' of w', is also a future state of w.
- This implies that we have to impose that R is transitive, i.e., wRw' and $w'Rw''$ implies wRw''.

Properties of accessibility relation

Temporal logic

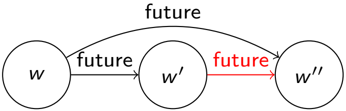


Properties of accessibility relation

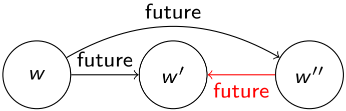
Depending on the intuitive interpretation of the accessibility relation between the possible worlds, we need to impose different properties on it. For instance:

Linear Temporal logic

- As in temporal logic, in linear modal temporal logics accessibility relation represents temporal relation between states of the world.
- but in addition we assume that there is only one future, i.e, we model exactly what happens, and not different possible futures.
- wRw' means that if we are in w sooner or later we will reach the state w'. i.e., w' is the future of w (and not one of the possible future states)
- this implies that in addiiton to transitivity we have to require totality on the order, i.e., if wRw' and wRw'' then either $w'Rw''$ or $w''Rw'$.



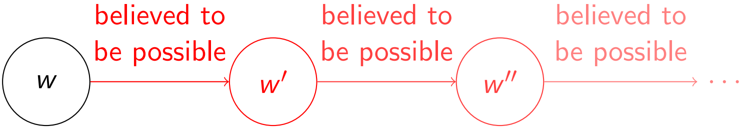
or



Properties of the accessibility relation

Logic of agent's beliefs

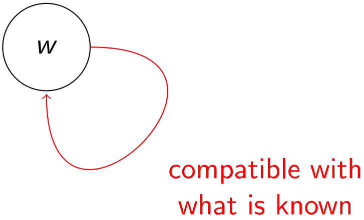
- if the accessibility relation is used to represent the knowledge of an agent A, and wRw' represents the fact that w' is a state of affairs that is believed to be possible by an agent.
- then it makes sense to assume that agents are "rational" in the sense that their beliefs are coherent and consistent, i.e., there is at least a state of affairs in which all their beliefs are true
- this corresponds to the property of R to be serial, i.e., for every world w there exists always a world w' which is accessible from w, i.e., wRw'.



Properties of the accessibility relation

Logic of agent's knowledge

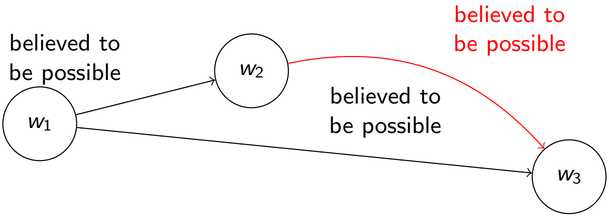
- a way to define knowledge is to say that it is true beliefs.
- If the accessibility relation wRw' represents the fact that w' is among the state of affairs that are compatible with what is known by an agent at state w
- that since what is known by an agent must be necessarily true in w, then w is compatible with what is known by the agent
- which implies that R is reflexive, i.e., wRw always holds.



Properties of the accessibility relation

Logic of agent's beliefs + negative introspection

- if an agent is suppose to know what he does not believe, i.e., he is conscious of the fact that he disbelieve something.
- then, if w' is a world that he considers to be impossible according to his beliefs in w, and w'' is a world that he considers possible according to his beliefs at w,
- then in w'' he will consider w' to be impossible
- this corresponds to the property of R to be euclidean, i.e. wRw'' and wRw' implies that $w'Rw''$.



Typical Properties of R

The following table summarizes the most relevant properties of the accessibility relation, which have been studied in modal logic, and for which it has been provided a sound and complete axiomatization

Properties of R

R is reflexive	$\forall w.R(w,w)$
R is transitive	$\forall w\ v\ u.(R(w,v) \wedge R(v,u) \supset R(w,u))$
R is symmetric	$\forall w\ v.(R(w,v) \supset R(v,w))$
R is euclidean	$\forall w\ v\ u.(R(w,v) \wedge R(w,u) \supset R(v,u))$
R is serial	$\forall w.\exists v R(w,v)$
R is functional	$\forall w \exists! v.R(w,v)$

This is not a complete list. There are other properties which we will not consider in this introductory part.

Properties of Accessibility relation - exercises

Exercise

Consider the formula $(B) = p \supset \Box\Diamond p$ Show that (B) is valid in a frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{R} \rangle$ if and only if $\mathcal{R}$ is symmetric.

Exercise

Consider the formula $(D) = \Box p \supset \Diamond p$ Show that (B) is valid in a frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{R} \rangle$ if and only if $\mathcal{R}$ is serial.

Exercise

Consider a modal language with two modalities $\Box_1$ and $\Box_2$. Let $\mathcal{F}$ be such: $\models_{\mathcal{F}} p \supset \Box_2\Diamond_2 p$. Can you tell which is the characteristic property of $\mathcal{F}$?

R is reflexive

The axiom T

If a frame is reflexive (we say that a frame has a property, when the relation R has such a property) then the formulas

$$\mathbf{T} \quad \Box\phi \supset \phi$$

holds. (Or alternatively $\phi \supset \Diamond\phi$.)

Discussion

- **T** is not valid ($\not\models$ **T**). Indeed conisder the frame composed of a signle world $F(W = \{w_0\}, R = \emptyset)$ and the model $M = (F, V)$ with $V(p) = \emptyset$. We ahve that $M, w_0 \models \Box p$, since there are no possible worlds accessible from w_0, but $M, w_0 \not\models p$, This implies that $M, w_0 \not\models \Box p \supset p$.
- **T** is valid in all the frames where R is reflexive ($\models_{\mathrm{Refl}}$ **T**). Suppose that $M, w \models \Box\phi$, this means that for all w' with wRw' $M, w' \models \phi$. Since wRw (reflexivity of R), $M, w \models \phi$. Which implies that $M, w \models \Box\phi \supset \phi$.

R is transitive

The axiom 4

If a frame is transitive then the formula

$$\mathbf{4} \quad \Box\phi \supset \Box\Box\phi$$

holds.

Discussion

- **4** is not valid ($\not\models$ **4**). Left as an exercise.
- **4** is valid in all the frames where R is transitive. Left as an exercise.

R is symmetric

The axiom B

If a frame is symmetric then the formula

$$\mathbf{B} \quad \phi \supset \Box\Diamond\phi$$

holds.

Discussion

- **B** is not valid ($\not\models$ **B**). Left as an exercise.
- **B** is valid in all the frames where R is symmetric. Left as an exercise.

R is serial

The axiom D

If a frame is serial then the formula

$$\textbf{D} \quad \Box\phi \supset \Diamond\phi$$

holds.

Discussion

- **D** is not valid ($\not\models$ **D**). Left as an exercise.
- **D** is valid in all the frames where R is serial. Left as an exercise.

Hilbert-Style Axiomatization of normal modal logics

Q: given a class of frames $\mathcal{C}$, are there syntactic mechanisms capable of generating the formulas valid on $\mathcal{C}$?

A: the Hilbert axiomatization of normal systems

Hilbert-Style Axiomatization of the class of Kripke frames (K)

Hilbert-Style Axiomatization – Axiom Schemata

PL: all instances of propositional tautologies, i.e., all formulas obtained from a propositional tautology ϕ by replacing every propositional letter p of ϕ with some modal formula. (e.g., $\Box\phi \vee \neg\Box\phi$ is obtained by replacing p with $\Box\phi$ in the propositional tautology $p \vee \neg p$)

Dual: $\Diamond\phi \supset \neg\Box\neg\phi$

K: $\Box(\phi \supset \psi) \supset (\Box\phi \supset \Box\psi)$

MP: $\dfrac{\phi \quad \phi \supset \psi}{\psi}$

Nec: $\dfrac{\phi}{\Box\phi}$

Remarks on Axiomatic Schemata

- **PL:** the starting point for modal reasoning
- **K:**
  - distribution axiom: the distribution of $\Box$ operator over $\supset$ operator
  - transform $\Box(\phi \supset \psi)$ into $\Box\phi \supset \Box\psi$
  - valid in all Kripke models
  - alethic reading: if it is necessary that ϕ implies ψ and ϕ is necessarily true then ψ is also necessary true
  - epistemic reading: if an (ideal) agent knows that ϕ implies ψ and knows ϕ, then he also knows ϕ. In other words the knowledge of agents is closed under logical consequence (rational agents)
- **Nec:**
  - allows to introduce $\Box$ operator in the proved formulas
  - alethic reading: if something is valid, then it is necessarily true
  - epistemic reading: agents know all the valid formulas (omniscient agents)

Additional axiom schemata

Axiom schema that captures properties of Frames

D: $\Box\phi \supset \Diamond\phi$

T: $\Box\phi \supset \phi$

B: $\phi \supset \Box\Diamond\phi$

4: $\Box\phi \supset \Box\Box\phi$

5: $\Diamond\phi \supset \Box\Diamond\phi$

Remarks on Axiom Schemata

D:
- alethic reading: if something is necessary, then it is possible
- deontic reading: if something is obligatory, then it is permitted

T:
- alethic reading: if something is necessary, then it is actually true
- epistemic reading: what is known is true (verity of knowledge)
  - knowledge axiom or truth axiom
  - distinguished feature of knowledge from belief

B: what is actually true is necessarily possible

4:
- epistemic reading: if you know something, then you know that you know it (positive introspection)

5:
- it is equivalent to $\neg\Box\phi \supset \Box\neg\Box\phi$
- epistemic reading: if you dont know something, then you know that you dont know it (negative introspection axiom)

Normal Systems of Modal Logic

- the minimal normal system **K**: PL+Dual+K+MP+Nec
- Lemmon code for normal systems: $\mathbf{KX_0}\dots\mathbf{X_m}$ denotes the system **K** plus axiomatic schemata $\mathbf{X_0},\dots,\mathbf{X_m}$
- some well-known systems
  - **KT**=**T**: the Gödel/Feys/Von Wright system
  - **KT4**=**S4**
  - **KT4B**=**KT45**=**S5**: the epistemic system
  - **KD**: deontic **T**
  - **KD4**: deontic **S4**
  - **KD45**: deontic **S5** or doxastic system
  - **KTB**: the Brouwer system

Proof Theory: Basic Notions

Definition

S-proof Let **S** be a normal system, and ϕ a wff. An S-proof is a finite sequence of wffs, each of which is an instance of an axiom schema in **S**, or follows from one or more earlier items in the sequence by applying a rule of inference

- $\vdash_{\mathbf{S}} \phi$ means that there is an **S**-proof $\phi_0, \dots, \phi_n$ such that $\phi = \phi_n$.
- If $\vdash_{\mathbf{S}} \phi$ we say that ϕ is a theorem of **S**