Mathematical logic

– 1^{st} assessment – Propositional Logic –

23 October 2013

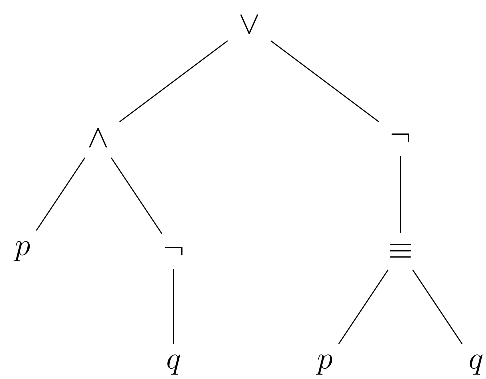
Exercise 1. [3 points] Consider the following formula

$$(p \wedge \neg q) \vee \neg(p \equiv q)$$

1. Write the formula as a tree, and
2. list all its sub-formulae

Solution.

1.



2.
   - $(p \wedge \neg q) \vee \neg(p \equiv q)$
   - $p \wedge \neg q$
   - p
   - $\neg q$
   - q
   - $\neg(p \equiv q)$
   - $p \equiv q$

Exercise 2. [6 points]:

1. Translate the following natural language sentences into propositional logic formulas:
   (a) Claudia gets a pay rise if she acquires a new customer or if she acquires a new project
   (b) Claudia does not acquire a new customer, however she gets a pay rise
   (c) Claudia acquires a new project
2. say whether (c) is a logical consequence of (a) and (b) using the truth tables and motivate your answer.

Solution.

1. Let

   R = Claudia gets a pay rise
   C = Claudia acquires a new customer
   P = Claudia acquires a new project

   A possible formalization is the following:

   (a) $(C \vee P) \supset R$
   (b) $\neg C \wedge R$
   (c) P

2. The truth table for the formulae above is:

		(c)		(b)		(a)
R	C	P	$\neg C$	$\neg C \wedge R$	$C \vee P$	$(C \vee P) \supset R$
T	T	T	F	F	T	T
T	T	F	F	F	T	T
T	F	T	T	T	T	T
T	F	F	T	T	F	T
F	T	T	F	F	T	F
F	T	F	F	F	T	F
F	F	T	T	F	T	F
F	F	F	T	F	F	T

As we can see from this truth table there are only two assignments that satisfy both premises: the ones in row 3 and 4. One of them (row 4) satisfies both (a) and (b) but does not satisfy (c). Therefore (c) is not a logical consequence of (a) and (b).

Exercise 3. [6 points] Let

(**MyRule1**) $$\frac{\phi \vee \psi \quad \neg\phi}{\psi}$$

and

(**MyRule2**) $$\frac{\phi \supset \psi \quad \neg\phi}{\neg\psi}$$

be two reasoning rules used to build proofs. Say (and prove) whether (**MyRule1**) and (**MyRule2**) are rules that preserve validity (i.e, that transform valid formulae in valid formulae).

Solution.

Proof.

- Let us consider (**MyRule1**).

  We have to prove that if $\phi \vee \psi$ and $\neg\phi$ are valid formulae, then ψ is a valid formula.

  Let us assume that $\phi \vee \psi$ and $\neg\phi$ are valid formulae. Then for each propositional interpretation $\mathcal{I}$ we have that $\mathcal{I} \models \phi \vee \psi$ and $\mathcal{I} \models \neg\phi$. From the definition of satisfiability of $\vee$, we have that $\mathcal{I} \models \phi$ or $\mathcal{I} \models \psi$. Since $\mathcal{I} \models \neg\phi$, and therefore $\mathcal{I} \not\models \phi$, then we can conclude that $\mathcal{I} \models \psi$. Thus ψ is a valid formula and (**MyRule1**) is a sound rule which preserves validity.

- Let us consider (**MyRule2**). Let p, q two propositional atoms, and let

  - $\phi = p \wedge \neg p$
  - $\psi = q$

  It is easy to prove that both $(p \wedge \neg p) \supset q$, and $\neg(p \wedge \neg p)$ are valid formulae. (can be done with the truth tables for instance)

  If we use them with the rule (**MyRule2**) we obtain $\neg q$ which is not a valid formula.

  Indeed let $\mathcal{I}$ be the interpretation $\mathcal{I} = q$. This interpretation does not satisfy $\neg q$.

  Thus rule (**MyRule2**) does not preserve validity.

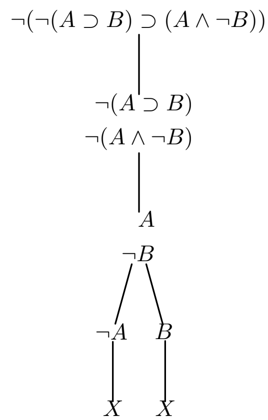
$\square$

Exercise 4. [6 points] For each of the following formulae determine whether they are **valid**, **unsatisfiable**, or **satisfiable** (and not valid) using analytic tableaux. Report the tableau, and use it to justify your answer.

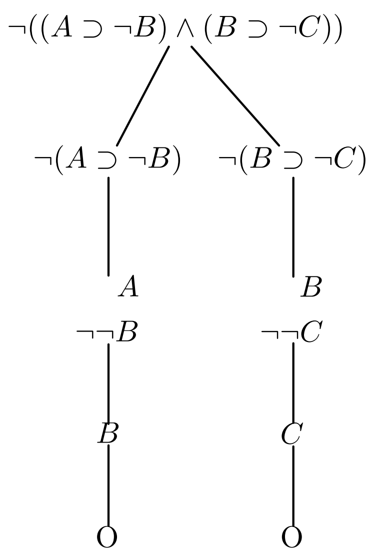
1. $\neg(A \supset B) \supset (A \wedge \neg B)$
2. $(A \supset \neg B) \wedge (B \supset \neg C)$

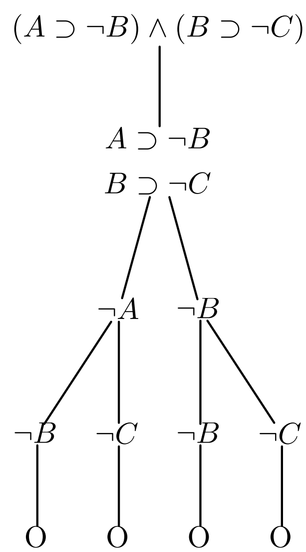
Solution.

1. The formula $\neg(A \supset B) \supset (A \wedge \neg B)$ is Valid. In fact, the tableau for its negated version is the closed tableau reported below



2. $(A \supset \neg B) \wedge (B \supset \neg C)$ is Satisfiable (and not valid). In fact, it is not valid, as shown by the tableau on the left hand side below, which remains open, and it is satisfiable, as shown by the tableau on the right hand side, which has four open branches and therefore shows at least four interpretations that make the formula true.





Exercise 5. [**3 points**] Apply DPLL procedure to check if the following set of clauses is satisfiable, and if it is so, return a partial assignment that makes the fomula true.

$$\phi = \{\{A, \neg B, \neg D\}, \{\neg A, \neg B, \neg C\}, \{\neg A, C, \neg D\}, \{\neg A, B, C\}\}$$

In the solution you have to specify all the application of unit propagation rule, and all the choices you take when Unit propagation is not applicable.

Solution.

1. ϕ does not contain unit clause, which implies that unit propagation is not applicable.
2. therefore, we select a literal (say A) and set $\mathcal{I}(A) = true$
3. Compute $\phi|_A$:
$$\phi|_A = \{\{\neg B, \neg C\}, \{C, \neg D\}, \{B, C\}\}$$
4. $\phi|_A$ does not contain unit clauses, therefore unit propagation is not applicable.
5. select a second literal, say $\neg B$, and set $\mathcal{I}(B) = false$
6. Compute $(\phi|_A)|_{\neg B}$ (also denoted by $\phi|_{A,\neg B}$).
$$\phi|_{A,\neg B} = \{\{C, \neg D\}, \{C\}\}$$
7. $\phi|_{A,\neg B}$ contain the unit clause $\{C\}$, we therefore extend the partial interpretation with $\mathcal{I}(C) = True$. We then apply unit propagation with $\{C\}$ as unit clause, obtaining $\phi|_{A,\neg B,C} = \{\}$, the empty set of clauses. Which means that the initial formula is satisfiable. The partial assignment is $\mathcal{I}(A) = True$, $\mathcal{I}(B) = false$ and $\mathcal{I}(C) = true$

Exercise 6. [**3 points**] Say when a formula ϕ is equi-satisfiable of a formula ψ. and show that the two formulas:

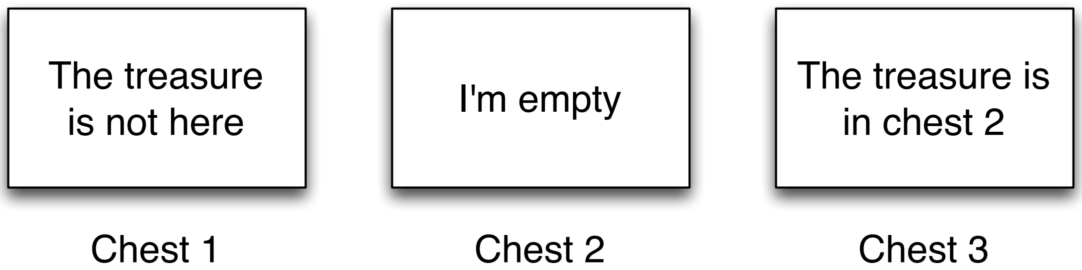
$$\phi = A \to (B \vee C) \qquad \psi = (N \equiv (B \vee C)) \wedge (A \to N) \tag{3}$$

are equi-satisfiable.

Solution. ϕ and ψ are equi-satisfiable, if and only if ϕ is satisfiable iff ψ is satisfiable. Or in other words, there is an interpretation $\mathcal{I}$ that satisfies ϕ if and only if there is an interpretation $\mathcal{J}$ that satisfies ψ.

Let us shows that the formulas in (3) are equisatisfiable. Let $\mathcal{I} \models \phi$. Let's extend $\mathcal{I}$ to $\mathcal{I}'$ setting $\mathcal{I}'(N) = \mathcal{I}(B \vee C)$. We have that $\mathcal{I}' \models \psi$. Viceersa, let $\mathcal{I}$ be an interpretation that satisfies ψ, then $\mathcal{I} \models N \equiv (B \vee C)$ implies that $\mathcal{I}(N) = \mathcal{I}(B \vee C)$. The fact that $\mathcal{I} \models A \to N$ impies that $\mathcal{I} \models A \to (B \vee C)$.

Exercise 7. [**6 points**] In her travels for treasure hunting, Chiara finds herself in front of three mysterious chests. In one of the chests is a fabulous treasure, all the others are empty. On each chest there is an inscription:



Given the fact that two chests are lying, and one is telling the truth, where is the treasure?

Solution. Let us define the following language:

- $t1$ = the treasure is in chest 1;
- $t2$ = the treasure is in chest 2;
- $t3$ = the treasure is in chest 3;

we can encode the knowledge we have as follows:

(a) "In one of the chests is a fabulous treasure, all the others are empty"

$$(t1 \wedge \neg t2 \wedge \neg t3) \vee (\neg t1 \wedge t2 \wedge \neg t3) \vee (\neg t1 \wedge \neg t2 \wedge t3)$$

(b) the sentence of chest 1: "the treasure is not here"

$$\neg t1$$

(c) the sentence of chest 2: "I'm empty"

$$\neg t2$$

(d) the sentence of chest 3: "the treasure is in chest 2"

$$t2$$

(e) "two chests are lying and one is telling the truth".

$$(\neg t1 \wedge \neg\neg t2 \wedge \neg t2) \vee (\neg\neg t1 \wedge \neg t2 \wedge \neg t2) \vee (\neg\neg t1 \wedge \neg\neg t2 \wedge t2)$$

This sentence can be simplified as follows:

$$(t1 \wedge \neg t2) \vee (t1 \wedge t2)$$

In building the truth tables for $t1$–$t3$ we can consider the combinations in which exactly one is true, to satisfy item (a). We find that row 1 is the only one that satisfies the sentences inscribed on all chests and also the requirement that one chest is telling the truth and two are lying. This row tells us that the treasure is in chest 1.

			(b)	(c)	(d)			(e)
$t1$	$t2$	$t3$	$\neg t1$	$\neg t2$	$t2$	$t1 \wedge \neg t2$	$t1 \wedge t2$	$(t1 \wedge \neg t2) \vee (t1 \wedge t2)$
T	F	F	F	T	F	T	F	T
F	T	F	T	F	T	F	F	F
F	F	T	T	T	F	F	F	F