Mathematical Logic Exam

10 June 2014

Instructions

- Answer in English and write in ink unless the question paper gives other instructions.
- Write clearly; illegible answers will not be marked.
- Take care to identify each answer clearly with:
  - the number of the exercise.
  - where appropriate, the part of the exercise you are answering.
- Clearly cross out rough working, or unwanted answers before handing in your answers.

Propositional Logic

Exercise 1 (PL Theory)**.** [**6 points**] Show that if both $\Gamma \cup \{\neg\phi\}$ and $\Gamma \cup \{\phi\}$ are not satisfiable then Γ is also not satisfiable.

Solution. *By contradiction, suppose that Γ is satisfiable, then there is an interpretation $\mathcal{I}$ that satisfies Γ, i.e., $\mathcal{I} \models \Gamma$. By definition of satisfiability in classical propositional logic, either $\mathcal{I} \models \phi$ or $\mathcal{I} \models \neg\phi$, i.e., that one of the two sets is satisfiable. But this contraddicts the fact that both sets $\Gamma \cup \{\phi\}$ and $\Gamma \cup \{\neg\phi\}$ are unsatisfiable.*

Exercise 2 (PL modeling)**.** [**6 points**] Alice and Bob are playing with a two face coin. In a first round each of them tosses the coin obtaining the same result. In a second round, the result of Alice toss is different from that of Bob. Show by means of truth tables that either Alice or Bob has obtained the same result in the two rounds.

Sugestion: Use the propositional letters A_1, A_2, B_1 and B_2 to represent the outcome of Alice and Bob tosses in the first and second round.

Solution.

- *Result first toss:* $A_1 \equiv B_1$
- *Result second toss:* $A_2 \equiv \neg B_2$

- *Alice same toss:* $A_1 \equiv A_2$
- *Bob same toss:* $B_1 \equiv B_2$

Show that the formula

$$A_1 \equiv B_1 \wedge A_2 \equiv \neg B_2 \supset A_1 \equiv A_1 \vee B_1 \equiv B_2$$

is valid

A_1	B_1	A_2	B_2	$(A_1 \equiv B_1$	$\wedge$	$A_2 \equiv \neg B_2)$	$\supset$	$(A_1 \equiv A_2$	$\vee$	$B_1 \equiv B_2)$
0	0	0	0	1	0	0	1	1	1	1
0	0	0	1	1	1	1	1	1	1	0
0	0	1	0	1	1	1	1	0	1	1
0	0	1	1	1	0	0	1	0	0	0
0	1	0	0	0	0	0	1	1	1	0
0	1	0	1	0	0	1	1	1	1	1
0	1	1	0	0	0	1	1	0	0	0
0	1	1	1	0	0	0	1	0	1	1
1	0	0	0	0	0	1	1	0	1	1
1	0	0	1	0	0	1	1	0	0	0
1	0	1	0	0	0	1	1	1	1	1
1	0	1	1	0	0	0	1	1	1	0
1	1	0	0	1	0	0	1	0	0	0
1	1	0	1	1	1	1	1	0	1	1
1	1	1	0	1	1	1	1	1	1	0
1	1	1	1	1	0	0	1	1	1	1

Exercise 3 (PL Reasoning)**.** [**6 points**] Prove by resolution that the formula $\neg Q \supset \neg R$ is not a logical consequence of the set of formulas $\{P \supset Q,\ \neg P \supset R\}$

Solution.

1. *To prove that a formula ϕ is not a logica consequence of a set of formulas Γ, we have to find a model for Γ and $\neg\phi$. I.e., we can check that $\Gamma \cup \{\neg\phi\}$ is satisfiable.*
2. *We therefore consider the three formulas*

$$P \supset Q$$
$$\neg P \supset R$$
$$\neg(Q \supset \neg R)$$

   and check via resolution if they are satisfiable

3. *We first transform the previous formulas in clausal normal form, obtaining:*

$$\{\neg P, Q\}$$
$$\{P, R\}$$
$$\{Q\}$$
$$\{R\}$$

4. *by applying resolution to the above formulas we can derive only the clause*

$$\{Q, R\}$$

and no other rules are applicable. This implies that from the initial set of formulas it is not possible to derive the empty clause. Which implies that the initial set of formulas are satisfiable.

Exercise 4 (FOL Theory). **[6 points]** Show that the tableaux rule is sound

$$\frac{\begin{array}{c}\vdots\\ \exists x \phi(x)\end{array}}{\phi(c)}$$

when c is a new constant, not appearing in the branch above $\exists x.\phi(x)$. Suggestion: you have to prove that if $\Gamma \cup \{\exists x.\phi(x)\}$ is satisfiable, then $\Gamma \cup \{\exists x.\phi(x),\ \phi(c)\}$ is also satisfiable. Explain why c must be <u>new</u>, i.e., that if it appears in Γ is it possible that the rule is not sound.

Solution.

A tableaux rule

$$\frac{\begin{array}{c}\vdots\\ \phi\end{array}}{\psi}$$

is sound whenever, if Γ is the set of formulas of in the branch above ϕ, and if $\Gamma \cup \{\phi\}$ is satisfiable, then $\Gamma \cup \{\phi, \psi\}$ is also satisfiable.

In this case, let Γ be the set of formulas occurring in the branch β above $\exists x.\phi(x)$. Suppose that $\Gamma \cup \{\exists x.\phi(x)\}$ is satisfiable. This means that there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \Gamma \cup \{\exists x.\phi(x)\}$. Therefore $\mathcal{I} \models \exists x.\phi(x)$. From the definition of satisfiability of $\exists x.\phi(x)$, we know that there is a $d \in \Delta^{\mathcal{I}}$ such that $\mathcal{I} \models \phi(x)[x := d]$. Let $\mathcal{I}'$ be the extension of $\mathcal{I}$, with $c^{\mathcal{I}'} = d$. This choice implies that $\mathcal{I}' \models \phi(c)$. Since c does not occur in any formulas of $\Gamma \cup \exists x.\phi(x)$, and $\mathcal{I}'$ coincides with $\mathcal{I}$ on the interpretation of all the other symbols, we have that $\mathcal{I}' \models \Gamma \cup \{\exists x.\phi(x)\}$. And therefore, we have that the formulas in the branch of $\phi(c)$, i.e., $\Gamma \cup \{\exists x.\phi(x), \phi(c)\}$ is satisfiable. Therefore we can conclude that the rule is sound.

Exercise 5 (FOL resolution). **[6 points]** Prove by resolution the validity of the following formula.

$$(\exists x \forall y.Q(x, y) \land \forall x.(Q(x, x) \supset \exists y.R(y, x))) \supset \exists y.\exists x.R(x, y)$$

Solution.

1. *negate the formula:*

$$\neg((\exists x \forall y.Q(x, y) \land \forall x.(Q(x, x) \supset \exists y.R(y, x))) \supset \exists y.\exists x.R(x, y))$$

2. *rename variables:*

$$\neg((\exists x \forall y.Q(x,y) \wedge \forall z.(Q(z,z) \supset \exists w.R(w,z))) \supset \exists v.\exists t.R(t,v))$$

3. *transform it in prenex normal form:*

$$\begin{array}{l}
(\exists x \forall y.Q(x,y) \wedge \forall z.(Q(z,z) \supset \exists w.R(w,z))) \wedge \neg\exists v.\exists t.R(t,v) \\
\exists x \forall y.Q(x,y) \wedge (\forall z.(\neg Q(z,z) \vee \exists w.R(w,z)) \wedge \forall v.\forall t.\neg R(t,v) \\
\exists x \forall y.Q(x,y) \wedge (\forall z.(\neg Q(z,z) \vee \exists w.R(w,z)) \wedge \forall v.\forall t.\neg R(t,v) \\
\exists x \forall y \forall z \exists w \forall v \forall t.(Q(x,y) \wedge (\neg Q(z,z) \vee R(w,z)) \wedge \neg R(t,v))
\end{array}$$

4. *Skolemize:*

$$\forall y \forall z \forall v \forall t.(Q(a,y) \wedge (\neg Q(z,z) \vee R(f(z),z)) \wedge \neg R(t,v))$$

5. *put in clausal form:*

$$\{Q(a,y)\},\ \{\neg Q(z,z), R(f(z),z)\}, \{\neg R(t,v)\}$$

6. *apply resolution and unification algorithm:*

$$\begin{array}{lll}
(1) & \{Q(a,y)\} & \textit{input clause} \\
(2) & \{\neg Q(z,z), R(f(z),z)\} & \textit{input clause} \\
(3) & \{\neg R(t,v)\} & \textit{input clause} \\
(4) & \{R(f(a),a)\} & \textit{from } (1) \textit{ and } (2) \textit{ with } \sigma = [a/z, z/y] \\
(5) & \{\} & \textit{from } (3) \textit{ and } (4) \textit{ with } \sigma = [f(a)/t, a/v]
\end{array}$$

Exercise 6 (Modal logics modelling). **[6 points]** Suppose you want to represent the preferences of Ana by the modal operator $\Box_{\text{Ana}}$ The formula $\Box_{\text{Ana}}\phi$ states that, in a certain situation, Ana prefers ϕ being true to ϕ being false.

For instance if the propositional variables R and H formalize the two propositions "it's raining" and "Ana stays at home", the formula $\Box_{\text{Ana}}H$ means that Alice prefers to stay at Home, while $\Box_{\text{Ana}}\neg R$ means that Alice prefers that it is not raining. The formula $\neg\Box_{\text{Ana}}R$ means that Alice does not prefer that it is raining. Notice that "non preferring something" is different from "preferring not something".

1. Using R and H and the modal operator $\Box_{\text{Ana}}$ formulate the following statements:
   (a) when it is raining Ana prefers to stay home
   (b) when it is not raining Ana has no preference between going out or staing at home.
2. Give a Kripke model that satisfies the formula $\neg\Box\phi \wedge \neg\Box\neg\phi$.

3. Use modal schemas to encode the following assumptions:

   (a) Ana prefers something that can be true. In other words, if a formula ϕ is always false then it cannot be preferred to $\neg\phi$ by Ana.

   (b) If Ana prefers $\phi \vee \psi$ to $\neg(\phi \vee \psi)$, then she either prefers ϕ over $\neg\phi$ or ψ over $\neg\psi$.

Solution.

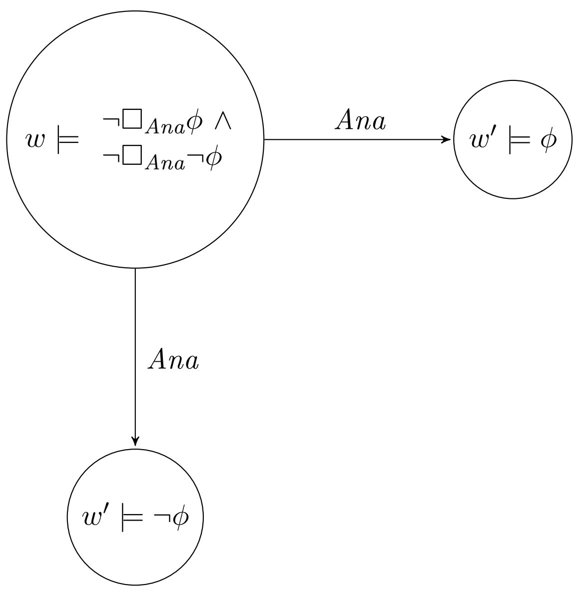
1. (a) when it is raining Ana prefers to stay home

$$R \supset \Box_{Ana} H$$

(b) when it is not raining Ana has no preference between going out or staing at home.

$$\neg R \supset \neg\Box_{Ana} H \wedge \neg\Box_{Ana}\neg H$$

2. A Kripke model that satisfies the formula $\neg\Box_{Ana}\phi \wedge \neg\Box_{Ana}\neg\phi$, should be a model that contains a world w where $\Box_{Ana}\phi$ and $\Box_{Ana}\neg\phi$ are both false. To force this ne need to have an accessible world where ϕ is false and one in which ϕ is true. The fact that $\Box_{Ana}\phi$ (resp. $\Box_{Ana}\neg\phi$) is true in w if and only if ϕ (res. $\neg\phi$) is true in all the worlds accessible from w, implies that if we have one world where ϕ is true and one where ϕ is false, the formulas $\Box_{Ana}\phi$ and $\Box_{Ana}\neg\phi$ both false.



3. Use modal schemas to encode the following assumptions:

(a) To represent the fact that if ϕ is always false then it cannot be preferred to $\neg\phi$ by Ana, we use the formula which is always false (i.e., $\bot$) and state that it is never preferred by Ana. As follows:

$$\neg\Box_{Ana}\bot$$

(b) If Ana prefers $\phi \vee \psi$ to $\neg(\phi \vee \psi)$, then she either prefers ϕ over $\neg\phi$ or ψ over $\neg\psi$.

$$\Box_{Ana}(\phi \vee \psi) \supset \Box_{Ana}\phi \vee \Box_{Ana}\psi$$