Express the following knowledge in first-order logic and add enough common sense statements (e.g. everyone has at most one spouse, nobody can be married to himself or herself, Tom, Sue and Mary are different people) to make it entail "Mary is not married" in first order logic.

Knowledge: There are exactly three people in the club, Tom, Sue and Mary. Tom and Sue are married. If a member of the club is married, their spouse is also in the club.

Answer We are going to use constants *tom*, *sue*, *mary*, unary predicate C for member of the club, and binary predicate $Married$.

Translation of the sentences:

S1 There are exactly three people in the club, Tom, Sue and Mary:

$$C(tom) \wedge C(sue) \wedge C(mary) \wedge \forall x(C(x) \supset (x = tom \vee x = sue \vee x = mary))$$

S2 Tom and Sue are married:

$$Married(tom, sue)$$

S3 If a member of the club is married, their spouse is also in the club:

$$\forall x \forall y(C(x) \wedge Married(x, y) \supset C(y))$$

S4 We need to show that this entails "Mary is not married":

$$\neg \exists y Married(mary, y)$$

In first-order logic, **S1-S3** do not entail **S4**. Consider for example an interpretation where Mary is married to Tom (nothing in **S1-S3** forbids this). Then **S1-S3** are true and **S4** is false.

We need to add some more sentences to **S1-S3** to make sure that **S4** is entailed:

S5 $\forall x \forall y(Married(x, y) \supset Married(y, x)$

S6 $\forall x \forall y \forall z(Married(x, y) \wedge Married(x, z) \supset y = z)$

S7 $\forall x \neg Married(x, x)$

S8 $\neg(tom = sue) \wedge \neg(tom = mary) \wedge \neg(mary = sue)$

Now we show that **S1-S3** together with **S5-S8** entail **S4**.

Suppose an interpretation $M = (D, I)$ satisfies **S1-S3** and **S5-S8** and does not satisfy **S4** (and we hope to derive a contradiction, that is something about M which is impossible).

Since M satisfies **S8**, we know that $I(tom)$, $I(sue)$ and $I(mary)$ are three different elements of D. Let us call them t, s and m. From **S2** we know that

$(t, s) \in I(Married)$. From **S5** we also know that $(s, t) \in I(Married)$ (consider an assignment which assigns t to x and s to y).

If **S4** is false, then there is some element $d \in D$ such that $(m, d) \in I(Married)$. From **S1** we know that $m \in I(C)$. So from **S3**, $d \in I(C)$ (consider an assignment which assigns m to x and d to y). From **S1** again, d must be equal to m, to t or to s. From **S7**, d cannot me equal to m. So it must be equal to t or to s.

If d is equal to t, then we have $(m, t) \in I(Married)$. By **S5**, $(t, m) \in I(Married)$. By **S6** and $(t, s) \in I(Married)$, this implies $s = m$ (consider an assignment which assigns t to x, s to y and m to z). But $s = m$ is false by **S8**, so d cannot be the same as t.

This means that the only option for d is to be s. Then $(m, s) \in I(Married)$ and by **S5** also $(s, m) \in I(Married)$. By **S6** and $(s, t) \in I(Married)$, this implies $t = m$ which is false.

So d cannot be any of t, s, m and cannot be any other element of D, which means that M satisfying all of **S1-S8** is impossible.