Mathematical Logic

First order logic: syntax and semantics

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Outline

- Why First Order Logic (FOL)?
- Syntax and Semantics of FOL;
- First Order Theories;
- ... and in between few examples;

Expressivity of propositional logic - I

Question

Try to express in Propositional Logic the following statements:

- Mary is a person
- John is a person
- Mary is mortal
- Mary and John are siblings

Expressivity of propositional logic - I

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A solution

Through atomic propositions:

- `Mary-is-a-person`
- `John-is-a-person`
- `Mary-is-mortal`
- `Mary-and-John-are-siblings`

Problem with previous solution

- `Mary-is-a-person`
- `John-is-a-person`
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How do we link `Mary` of the first sentence to `Mary` of the third sentence? Same with `John`. How do we link `Mary` and `Mary-and-John`?

Expressivity of propositional logic - II

Question

Try to express in Propositional Logic the following statements:

- All persons are mortal;
- There is a person who is a spy.

Expressivity of propositional logic - II

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Try to express in Propositional Logic the following statements:

- All persons are mortal;
- There is a person who is a spy.

A solution

We can give all people a name and express this fact through atomic propositions:

- $\texttt{Mary-is-mortal} \wedge \texttt{John-is-mortal} \wedge \texttt{Chris-is-mortal} \wedge \ldots \wedge \texttt{Michael-is-mortal}$
- $\texttt{Mary-is-a-spy} \vee \texttt{John-is-a-spy} \vee \texttt{Chris-is-a-spy} \vee \ldots \vee \texttt{Michael-is-a-spy}$

Problem with previous solution

- $\texttt{Mary-is-mortal} \wedge \texttt{John-is-mortal} \wedge \texttt{Chris-is-mortal} \wedge \ldots \wedge \texttt{Michael-is-mortal}$
- $\texttt{Mary-is-a-spy} \vee \texttt{John-is-a-spy} \vee \texttt{Chris-is-a-spy} \vee \ldots \vee \texttt{Michael-is-a-spy}$

Problem with previous solution

- Mary-is-mortal $\wedge$ John-is-mortal $\wedge$ Chris-is-mortal $\wedge \ldots \wedge$ Michael-is-mortal
- Mary-is-a-spy $\vee$John-is-a-spy $\vee$Chris-is-a-spy $\vee \ldots \vee$ Michael-is-a-spy

The representation is not compact and generalization patterns are difficult to express.

Problem with previous solution

- Mary-is-mortal $\wedge$ John-is-mortal $\wedge$ Chris-is-mortal $\wedge \ldots \wedge$ Michael-is-mortal
- Mary-is-a-spy $\vee$John-is-a-spy $\vee$Chris-is-a-spy $\vee \ldots \vee$ Michael-is-a-spy

The representation is not compact and generalization patterns are difficult to express.
What is we do not know all the people in our "universe"? How can we express the statement independently from the people in the "universe"?

Expressivity of propositional logic - III

Question

Try to express in Propositional Logic the following statements:

- Every natural number is either even or odd

Expressivity of propositional logic - III

Question

Try to express in Propositional Logic the following statements:

- Every natural number is either even or odd

A solution

We can use two families of propositions $even_i$ and odd_i for every $i \geq 1$, and use the set of formulas

$$\{odd_i \vee even_i | i \geq 1\}$$

Problem with previous solution

$$\{odd_i \vee even_i | i \geq 1\}$$

What happens if we want to state this in one single formula? To do this we would need to write an infinite formula like:

$$(odd_1 \vee even_1) \wedge (odd_2 \vee even_2) \wedge \ldots$$

and this cannot be done in propositional logic.

Expressivity of propositional logic -IV

Question

Express the statements:

- the father of Luca is Italian

Solution (Partial)

- `mario-is-father-of-luca` $\supset$ `mario-is-italian`
- `michele-is-father-of-luca` $\supset$ `michele-is-italian`
- ...

Problem with previous solution

- `mario-is-father-of-luca` $\supset$ `mario-is-italian`
- `michele-is-father-of-luca` $\supset$ `michele-is-italian`
- ...

This statement strictly depend from a fixed set of people. What happens if we want to make this statement independently of the set of persons we have in our universe?

Why first order logic?

Because it provides a way of representing information like the following one:

1. Mary is a person;
2. John is a person;
3. Mary is mortal;
4. Mary and John are siblings
5. Every person is mortal;
6. There is a person who is a spy;
7. Every natural number is either even or odd;
8. The father of Luca is Italian

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and also to infer the third one from the first one and the fifth one.

First order logic

Whereas propositional logic assumes world contains facts, first-order logic (like natural language) assumes the world contains:

- Constants: mary, john, 1, 2, 3, red, blue, world war 1, world war 2, 18th Century...
- Predicates: Mortal, Round, Prime, Brother of, Bigger than, Inside, Part of, Has color, Occurred after, Owns, Comes between, ...
- Functions: Father of, Best friend, Third inning of, One more than, End of, ...

- Mary is a person
- John is a person
- Mary is mortal
- Mary and John are siblings

In FOL it is possible to build an atomic propositions by applying a predicate to constants

- $Person(mary)$
- $Person(john)$
- $Mortal(mary)$
- $Siblings(mary, john)$

Quantifiers and variables

- Every person is mortal;
- There is a person who is a spy;
- Every natural number is either even or odd;

In FOL it is possible to build propositions by applying universal (existential) quantifiers to variables. This allows to quantify to arbitrary objects of the universe.

- $\forall x.Person(x) \supset Mortal(x)$;
- $\exists x.Person(x) \supset Spy(x)$;
- $\forall x.(Odd(x) \vee Even(x))$

Functions

- The father of Luca is Italian.

In FOL it is possible to build propositions by applying a function to a constant, and then a predicate to the resulting object.

- $Italian(fatherOf(Mario))$

Syntax of FOL

Logical symbols

- the logical constant $\bot$
- propositional logical connectives $\wedge$, $\vee$, $\supset$, $\neg$, $\equiv$
- the quantifiers $\forall$, $\exists$
- an infinite set of variable symbols $x_1, x_2, \ldots$
- the equality symbol $=$. (optional)

Non Logical symbols

- a set $c_1, c_2, \ldots$ of constant symbols
- a set $f_1, f_2, \ldots$ of functional symbols each of which is associated with its *arity* (i.e., number of arguments)
- a set $P_1, P_2, \ldots$ of *relational symbols* each of which is associated with its *arity* (i.e., number of arguments)

Terms and formulas of FOL

Terms

- every constant c_i and every variable x_i is a term;
- if $t_1, \ldots, t_n$ are terms and f_i is a functional symbol of arity equal to n, then $f(t_1, \ldots, t_n)$ is a term

Well formed formulas

- if t_1 and t_2 are terms then $t_1 = t_2$ is a formula
- If $t_1, \ldots, t_n$ are terms and P_i is relational symbol of arity equal to n, then $P_i(t_1, \ldots, t_n)$ is formula
- if A and B are formulas then $\bot$, $A \wedge B$, $A \supset B$, $A \vee B$ $\neg A$ are formulas
- if A is a formula and x a variable, then $\forall x.A$ and $\exists x.A$ are formulas.

Examples of terms and formulas

Example (Terms)

- x_i,
- c_i,
- $f_i(x_j, c_k)$, and
- $f(g(x, y), h(x, y, z), y)$

Example (formulas)

- $f(a, b) = c$,
- $P(c_1)$,
- $\exists x(A(x) \vee B(y))$, and
- $P(x) \supset \exists y.Q(x, y)$.

An example of representation in FOL

Example (Language)

constants	functions (arity)	Predicate (arity)
Aldo	mark (2)	attend (2)
Bruno	best-friend (1)	friend (2)
Carlo		student (1)
MathLogic		course (1)
DataBase		less-than (2)
0, 1, . . . , 10		

Example (Terms)

Intuitive meaning
an individual named Aldo
the mark 1
Bruno's best friend
anything
Bruno's mark in MathLogic
somebody's mark in DataBase
Bruno's best friend mark in MathLogic

An example of representation in FOL

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Example (Terms)

Intuitive meaning	term
an individual named Aldo	Aldo
the mark 1	1
Bruno's best friend	best-friend(Bruno)
anything	x
Bruno's mark in MathLogic	mark(Bruno,MathLogic)
somebody's mark in DataBase	mark(x,DataBase)
Bruno's best friend mark in MathLogic	mark(best-friend(Bruno),MathLogic)

An example of representation in FOL (cont'd)

Example (Formulas)

Intuitive meaning
Aldo and Bruno are the same person
Carlo is a person and MathLogic is a course
Aldo attends MathLogic
Courses are attended only by students
every course is attended by somebody
every student attends something
a student who attends all the courses
every course has at least two attenders
Aldo's best friend attend the same courses attended by Aldo
best-friend is symmetric
Aldo and his best friend have the same mark in MathLogic
A student can attend at most two courses

An example of representation in FOL (cont'd)

Example (Formulas)

Intuitive meaning	Formula
Aldo and Bruno are the same person	$Aldo = Bruno$
Carlo is a person and MathLogic is a course	$person(Carlo) \wedge course(MathLogic)$
Aldo attends MathLogic	$attend(Aldo, MathLogic)$
Courses are attended only by students	$\forall x(attend(x, y) \supset course(y) \supset student(x))$
every course is attended by somebody	$\forall x(course(x) \supset \exists y\ attend(y, x))$
every student attends something	$\forall x(student(x) \supset \exists y\ attend(x, y))$
a student who attends all the courses	$\exists x(student(x) \wedge \forall y(course(y) \supset attend(x, y)))$
every course has at least two attenders	$\forall x(course(x) \supset \exists y \exists z(attend(y, x) \wedge attend(z, x) \wedge \neg y = z))$
Aldo's best friend attend the same courses attended by Aldo	$\forall x(attend(Aldo, x) \supset attend(best\text{-}friend(Aldo), x))$
best-friend is symmetric	$\forall x(best\text{-}friend(best\text{-}friend(x)) = x)$
Aldo and his best friend have the same mark in MathLogic	$mark(\text{best-friend}(Aldo), MathLogic) = mark(Aldo, MathLogic)$
A student can attend at most two courses	$\forall x \forall y \forall z \forall w(attend(x, y) \wedge attend(x, z) \wedge attend(x, w) \supset (y = z \vee z = w \vee y = w))$

Common Mistakes

- Use of $\wedge$ with $\forall$

$$\forall x\ (WorksAt(FBK, x) \wedge Smart(x))$$

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  $\forall x \; (WorksAt(FBK, x) \wedge Smart(x))$ means "Everyone works at FBK and everyone is smart"

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- Use of $\supset$ with $\exists$

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- Use of $\supset$ with $\exists$

  $\exists x\ (WorksAt(FBK, x) \supset Smart(x))$ mans "There is a person so that if (s)he works at FBK then (s)he is smart" and this is true as soon as there is at last an x who does not work at FBK

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  "Everyone working at FBK is smart" is formalized as $\forall x\ (WorksAt(FBK, x) \supset Smart(x))$

- Use of $\supset$ with $\exists$

  $\exists x\ (WorksAt(FBK, x) \supset Smart(x))$ mans "There is a person so that if (s)he works at FBK then (s)he is smart" and this is true as soon as there is at last an x who does not work at FBK

  "There is an FBK-working smart person" is formalized as $\exists x\ (WorksAt(FBK, x) \wedge Smart(x))$

Representing variations of quantifiers in FOL

Example

Represent the statement at most 2 students attend the KR course

$$\forall x_1 \forall x_2 \forall x_3 (attend(x_1, KR) \wedge attend(x_2, KR) \wedge attend(x_2, KR) \supset x_1 = x_2 \vee x_2 = x_3 \vee x_1 = x_3)$$

At most n ...

$$\forall x_1 \ldots x_{n+1} \left(\bigwedge_{i=1}^{n+1} \phi(x_i) \supset \bigvee_{i \neq j=1}^{n+1} x_i = x_j \right)$$

Representing variations quantifiers in FOL

Example

Represent the statement at least 2 students attend the KR course

$$\exists x_1 \exists x_2 (attend(x_1, KR) \wedge attend(x_2, KR) \wedge x_1 \neq x_2)$$

At least n . . .

$$\exists x_1 \ldots x_n \left(\bigwedge_{i=1}^{n} \phi(x_i) \wedge \bigwedge_{i \neq j=1}^{n} x_i \neq x_j \right)$$

Semantics of FOL

FOL interpretation for a language L

A first order interpretation for the language $L = \langle c_1, c_2, \ldots, f_1, f_2, \ldots, P_1, P_2, \ldots \rangle$ is a pair $\langle \Delta, \mathcal{I} \rangle$ where

- Δ is a non empty set called interpretation domain
- $\mathcal{I}$ is is a function, called interpretation function
  - $\mathcal{I}(c_i) \in \Delta$ (elements of the domain)
  - $\mathcal{I}(f_i) : \Delta^n \to \Delta$ (n-ary function on the domain)
  - $\mathcal{I}(P_i) \subseteq \Delta^n$ (n-ary relation on the domain)

where n is the arity of f_i and P_i.

Example of interpretation

Example (Of interpretation)

Symbols

Constants: *alice*, *bob*, *carol*, *robert*
Function: *mother-of* (with arity equal to 1)
Predicate: *friends* (with arity equal to 2)

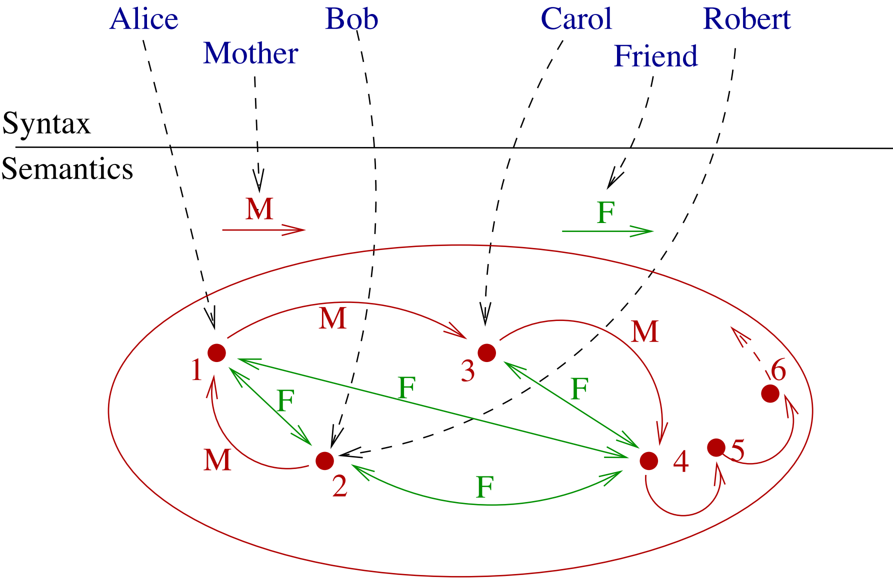
Domain $\Delta = \{1, 2, 3, 4, \dots\}$

Interpretation $\mathcal{I}(alice) = 1$, $\mathcal{I}(bob) = 2$, $\mathcal{I}(carol) = 3$, $\mathcal{I}(robert) = 2$

$$\mathcal{I}(mother\text{-}of) = M \quad \begin{array}{l} M(1) = 3 \\ M(2) = 1 \\ M(3) = 4 \\ M(n) = n + 1 \text{ for } n \geq 4 \end{array}$$

$$\mathcal{I}(friends) = F = \left\{ \begin{array}{lll} \langle 1, 2 \rangle, & \langle 2, 1 \rangle, & \langle 3, 4 \rangle, \\ \langle 4, 3 \rangle, & \langle 4, 2 \rangle, & \langle 2, 4 \rangle, \\ \langle 4, 1 \rangle, & \langle 1, 4 \rangle, & \langle 4, 4 \rangle \end{array} \right\}$$

Example (cont'd)



Definition (Assignment)

An assignment a is a function from the set of variables to Δ.

$a[x/d]$ denotes the assignment that coincides with a on all the variables but x, which is associated to d.

Definition (Interpretation of terms)

The interpretation of a term t w.r.t. the assignment a, in symbols $\mathcal{I}(t)[a]$ is recursively defined as follows:

$$
\begin{aligned}
\mathcal{I}(x_i)[a] &= a(x_i) \\
\mathcal{I}(c_i)[a] &= \mathcal{I}(c_i) \\
\mathcal{I}(f(t_1, \ldots, t_n))[a] &= \mathcal{I}(f)(\mathcal{I}(t_1)[a], \ldots, \mathcal{I}(t_n)[a])
\end{aligned}
$$

FOL Satisfiability of formulas

Definition (Satisfiability of a formula w.r.t. an assignment)

An interpretation $\mathcal{I}$ satisfies a formula ϕ w.r.t. the assignment a according to the following rules:

$$
\begin{array}{rcl}
\mathcal{I} \models t_1 = t_2[a] & \text{iff} & \mathcal{I}(t_1)[a] = \mathcal{I}(t_2)[a] \\
\mathcal{I} \models P(t_1, \dots, t_n)[a] & \text{iff} & \langle \mathcal{I}(t_1)[a], \dots, \mathcal{I}(t_n)[a] \rangle \in \mathcal{I}(P) \\
\mathcal{I} \models \phi \wedge \psi[a] & \text{iff} & \mathcal{I} \models \phi[a] \text{ and } \mathcal{I} \models \psi[a] \\
\mathcal{I} \models \phi \vee \psi[a] & \text{iff} & \mathcal{I} \models \phi[a] \text{ or } \mathcal{I} \models \psi[a] \\
\mathcal{I} \models \phi \supset \psi[a] & \text{iff} & \mathcal{I} \not\models \phi[a] \text{ or } \mathcal{I} \models \psi[a] \\
\mathcal{I} \models \neg\phi[a] & \text{iff} & \mathcal{I} \not\models \phi[a] \\
\mathcal{I} \models \phi \equiv \psi[a] & \text{iff} & \mathcal{I} \models \phi[a] \text{ iff } \mathcal{I} \models \psi[a] \\
\mathcal{I} \models \exists x\phi[a] & \text{iff} & \text{there is a } d \in \Delta \text{ such that } \mathcal{I} \models \phi[a[x/d]] \\
\mathcal{I} \models \forall x\phi[a] & \text{iff} & \text{for all } d \in \Delta, \mathcal{I} \models \phi[a[x/d]]
\end{array}
$$

Example (cont'd)

Exercise

Check the satisfiability of the following statements, considering the interpretation defined few slides ago:

1. $\mathcal{I} \models Alice = Bob[a]$
2. $\mathcal{I} \models Robert = Bob[a]$
3. $\mathcal{I} \models x = Bob[a[x/2]]$

Example (cont.)

$$\mathcal{I}(\textit{mother-of}(\textit{alice}))[a] = 3$$

$$\mathcal{I}(\textit{mother-of}(x))[a[x/4]] = 5$$

$\mathcal{I}(\textit{friends}(x, y)) =$

$x :=$	$y :=$
1	2
2	1
4	1
1	4
4	2
2	4
4	3
3	4
4	4

$\mathcal{I}(\textit{friends}(x, x)) =$

$x :=$
4

$\mathcal{I}(\textit{friends}(x, y) \wedge x = y) =$

$x :=$	$y :=$
4	4

$\mathcal{I}(\exists x \textit{friends}(x, y)) =$

$y :=$
2
1
4
3

$\mathcal{I}(\forall x \textit{friends}(x, y)) =$

$y :=$
4

Analogy with Databases

When the language $\mathcal{L}$ and the domain of interpretation Δ are finite, and $\mathcal{L}$ does not contains functional symbols (relational language), there is a strict analogy between first order logics and databases.

- Non logical simbols of $\mathcal{L}$ correspond to database schema (tables)
- Δ corresponds to the set of values which appears in the tables (active domain)
- the interpretation $\mathcal{I}$ corresponds to the tuples that belongs to each relation
- Formulas on $\mathcal{L}$ corresponds to query over the database
- Interpretation of formulas of $\mathcal{L}$ correspond to answers.

Analogy with Databases

FOL	DB
friends	`CREATE TABLE FRIENDS (friend1 : INTEGER` `friend2 : INTEGER)`
$friends(x, y)$	`SELECT * FROM FRIENDS`

$friends(x, x)$	`SELECT friend1`
	`FROM FRIENDS`
	`WHERE friends1 = friends2`

$friends(x, y) \wedge x = y$	`SELECT * FROM FRIENDS` `WHERE friends1 = friends2`
$\exists x.friends(x, y)$	`SELECT friend2` `FROM FRIENDS`

$friends(x, y) \wedge friends(y, z)$	`SELECT *`
	`FROM FRIENDS as FRIEND1`
	`FRIENDS as FRIEND2`
	`WHERE FRIENDS1.friends2 = FRIENDS2.friends1`

Free variables

Intuition

A free occurrence of a variable x is an occurrence of x which is not bounded by a (universal or existential) quantifier.

Definition (Free occurrence)

- any occurrence of x in t_k is free in $P(t_1, \ldots, t_k, \ldots, t_n)$
- any free occurrence of x in ϕ or in ψ is also fee in $\phi \wedge \psi$, $\psi \vee \phi$, $\psi \supset \phi$, and $\neg\phi$
- any free occurrence of x in ϕ, is free in $\forall y.\phi$ and $\exists y.\phi$ if y is distinct from x.

Definition (Ground/Closed Formula)

A formula ϕ is ground if it does not contain any variable. A formula is closed if it does not contain free occurrences of variables.

Free variables

A variable x is free in ϕ (denote by $\phi(x)$) if there is at least a free occurrence of x in ϕ.
Free variables represents individuals which must be instantiated to make the formula a meaningful proposition.

- x is free in $friends(alice, x)$.
- x is free in $P(x) \supset \forall x.Q(x)$ (the occurrence of x in red is free the one in green is not free.

Free variables - intuition

Intuitively..

Free variables represents individuals which must be instantiated to make the formula a meaningful proposition.

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- $Friends(Bob, y)$

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- $Friends(Bob, y)$ y free

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- $Friends(Bob, y)$ $\quad$ y free
- $\forall y.Friends(Bob, y)$

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Free variables represents individuals which must be instantiated to make the formula a meaningful proposition.

- $Friends(Bob, y)$ $\quad$ y free
- $\forall y.Friends(Bob, y)$ $\quad$ no free variables

Free variables - intuition

Intuitively..

Free variables represents individuals which must be instantiated to make the formula a meaningful proposition.

- $Friends(Bob, y)$ y free
- $\forall y.Friends(Bob, y)$ no free variables
- $Sum(x, 3) = 12$

Free variables - intuition

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- $Friends(Bob, y)$ y free
- $\forall y.Friends(Bob, y)$ no free variables
- $Sum(x, 3) = 12$ x free

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- $Friends(Bob, y)$ y free
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- $Sum(x, 3) = 12$ x free
- $\exists x.(Sum(x, 3) = 12)$

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- $\exists x.(Sum(x, y) = 12)$

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- $Sum(x, 3) = 12$ x free
- $\exists x.(Sum(x, 3) = 12)$ no free variables
- $\exists x.(Sum(x, y) = 12)$ y free

Free variable and free terms

Definition (Term free for a variable)

A term t is free for a variable x in formula ϕ, if and only if all the occurrences of x in ϕ do not occur within the scope of a quantifier of some variable occurring in t.

Example

The term x is free for y in $\exists z.hates(y,z)$. We can safely replace y with x obtaining $\exists z.hates(x,z)$ without changing the meaning of the formula.
However, the term z is not free for y in $\exists z.hates(y,z)$. In fact y occurs within the scope of a quantifier of z. Thus, we cannot substitute z for y in this sentence without changing the meaning of the sentence as we obtain $\exists z.hates(z,z)$.

Free variables and free terms - example

An occurrence of a variable x can be safely instantiated by a term free for x in a formula ϕ,
If you replace x with a terms which is not free for x in ϕ, you can have unexpected effects:
E.g., replacing x with $\mathit{mother\text{-}of}(y)$ in the formula $\exists y.\mathit{friends}(x, y)$ you obtain the formula

$$\exists y.\mathit{friends}(\mathit{mother\text{-}of}(y), y)$$

Satisfiability and Validity

Definition (Model, satisfiability and validity)

An interpretation $\mathcal{I}$ is a model of ϕ under the assignment a, if

$$\mathcal{I} \models \phi[a]$$

A formula ϕ is satisfiable if there is some $\mathcal{I}$ and some assignment a such that $\mathcal{I} \models \phi[a]$.
A formula ϕ is unsatisfiable if it is not satisfiable.
A formula ϕ is valid if every $\mathcal{I}$ and every assignment a $\mathcal{I} \models \phi[a]$

Definition (Logical Consequence)

A formula ϕ is a logical consequence of a set of formulas Γ, in symbols $\Gamma \models \phi$, if for all interpretations $\mathcal{I}$ and for all assignment a

$$\mathcal{I} \models \Gamma[a] \quad \Longrightarrow \quad \mathcal{I} \models \phi[a]$$

where $\mathcal{I} \models \Gamma[a]$ means that $\mathcal{I}$ satisfies all the formulas in Γ under a.

Excercises

Say where these formulas are valid, satisfiable, or unsatisfiable

- $\forall x P(x)$
- $\forall x P(x) \supset \exists y P(y)$
- $\forall x.\forall y.(P(x) \supset P(y))$
- $P(x) \supset \exists y P(y)$
- $P(x) \vee \neg P(y)$
- $P(x) \wedge \neg P(y)$
- $P(x) \supset \forall x.P(x)$
- $\forall x \exists y.Q(x,y) \supset \exists y \forall x Q(x,y)$
- $x = x$
- $\forall x.P(x) \equiv \forall y.P(y)$
- $x = y \supset \forall x.P(x) \equiv \forall y.P(y)$
- $x = y \supset (P(x) \equiv P(y))$
- $P(x) \equiv P(y) \supset x = y$

Properties of quantifiers

Proposition

The following formulas are valid

- $\forall x(\phi(x) \wedge \psi(x)) \equiv \forall x \phi(x) \wedge \forall x \psi(x)$
- $\exists x(\phi(x) \vee \psi(x)) \equiv \exists x \phi(x) \vee \exists x \psi(x)$
- $\forall x \phi(x) \equiv \neg \exists x \neg \phi(x)$
- $\forall x \exists x \phi(x) \equiv \exists x \phi(x)$
- $\exists x \forall x \phi(x) \equiv \forall x \phi(x)$

Proposition

The following formulas are not valid

- $\forall x(\phi(x) \vee \psi(x)) \equiv \forall x \phi(x) \vee \forall x \psi(x)$
- $\exists x(\phi(x) \wedge \psi(x)) \equiv \exists x \phi(x) \wedge \exists x \psi(x)$
- $\forall x \phi(x) \equiv \exists x \phi(x)$
- $\forall x \exists y \phi(x, y) \equiv \exists y \forall x \phi(x, y)$

Expressing properties in FOL

What is the meaning of the following FOL formulas?

1. $bought(Frank, dvd)$
2. $\exists x.bought(Frank, x)$
3. $\forall x.(bought(Frank, x) \rightarrow bought(Susan, x))$
4. $(\forall x.bought(Frank, x)) \rightarrow (\forall x.bought(Susan, x))$
5. $\forall x \exists y.bought(x, y)$
6. $\exists x \forall y.bought(x, y)$

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6. $\exists x \forall y.bought(x, y)$

1. "Frank bought a dvd."
2. "Frank bought something."
3. "Susan bought everything that Frank bought."
4. "If Frank bought everything, so did Susan."
5. "Everyone bought something."
6. "Someone bought everything."

Expressing properties in FOL

Define an appropriate language and formalize the following sentences using FOL formulas.

1. All Students are smart.
2. There exists a student.
3. There exists a smart student.
4. Every student loves some student.
5. Every student loves some other student.
6. There is a student who is loved by every other student.
7. Bill is a student.
8. Bill takes either Analysis or Geometry (but not both).
9. Bill takes Analysis and Geometry.
10. Bill doesn't take Analysis.
11. No students love Bill.

Expressing properties in FOL

1. $\forall x.(Student(x) \rightarrow Smart(x))$
2. $\exists x.Student(x)$
3. $\exists x.(Student(x) \wedge Smart(x))$
4. $\forall x.(Student(x) \rightarrow \exists y.(Student(y) \wedge Loves(x,y)))$
5. $\forall x.(Student(x) \rightarrow \exists y.(Student(y) \wedge \neg(x = y) \wedge Loves(x,y)))$
6. $\exists x.(Student(x) \wedge \forall y.(Student(y) \wedge \neg(x = y) \rightarrow Loves(y,x)))$
7. $Student(Bill)$
8. $Takes(Bill, Analysis) \leftrightarrow \neg Takes(Bill, Geometry)$
9. $Takes(Bill, Analysis) \wedge Takes(Bill, Geometry)$
10. $\neg Takes(Bill, Analysis)$
11. $\neg \exists x.(Student(x) \wedge Loves(x, Bill))$

Expressing properties in FOL

For each property write a formula expressing the property, and for each formula writhe the property it formalises.

- Every Man is Mortal
- Every Dog has a Tail
- There are two dogs
- Not every dog is white
- $\exists x.Dog(x) \wedge \exists y.Dog(y)$
- $\forall x, y(Dog(x) \wedge Dog(y) \supset x = y)$

Expressing properties in FOL

For each property write a formula expressing the property, and for each formula writhe the property it formalises.

- Every Man is Mortal
  $\forall x.Man(x) \supset Mortal(x)$
- Every Dog has a Tail
  $\forall x.Dog(x) \supset \exists y(PartOf(x, y) \wedge Tail(y))$
- There are two dogs
  $\exists x, y(Dog(x) \wedge Dog(y) \wedge x \neq y)$
- Not every dog is white
  $\neg\forall x.Dog(x) \supset White(x)$
- $\exists x.Dog(x) \wedge \exists y.Dog(y)$
  There is a dog
- $\forall x, y(Dog(x) \wedge Dog(y) \supset x = y)$
  There is at most one dog

Open and Closed Formulas

- Note that for closed formulas, satisfiability, validity and logical consequence do not depend on the assignment of variables.
- For closed formulas, we therefore omit the assignment and write $\mathcal{I} \models \phi$.
- More in general $\mathcal{I} \models \phi[a]$ if and only if $\mathcal{I} \models \phi[a']$ when $[a]$ and $[a']$ coincide on the variables free in ϕ (they can differ on all the others)

First order theories

- Mathematics focuses on the study of properties of certain structures. E.g. Natural/Rational/Real/Complex numbers, Algebras, Monoids, Lattices, Partially-ordered sets, Topological spaces, fields, ...
- In knowledge representation, mathematical structures can be used as a reference abstract model for a real world feature. e.g.,
  - natural/rational/real numbers can be used to represent linear time;
  - trees can be used to represent possible future evolutions;
  - graphs can be used to represent maps;
  - ...
- Logics provides a rigorous way to describe certain classes of mathematical structures.

First order theory

Definition (First order theory)

A first order theory is a set of formulas of the FOL language closed under the logical consequence relation. That is, T is a theory iff $T \models A$ implies that $A \in T$

Remark

A FOL theory always contains an infinite set of formulas. Indeed any theory T contains at least all the valid formulas (which are infinite).

Definition (Set of axioms for a theory)

A set of formulas Ω is a set of axioms for a theory T if for all $\phi \in T$, $\Omega \models \phi$.

First order theory (cont'd)

Definition

Finitely axiomatizable theory A theory T is finitely axiomatizable if it has a finite set of axioms.

Definition (Axiomatizable structure)

Given a class of mathematical structures C for a language L, we say that a theory T is a sound and complete axiomatization of C if and only if

$$T \models \phi \qquad \Longleftrightarrow \qquad \mathcal{I} \models \phi \ \text{ for all } \mathcal{I} \in C$$

Examples of first order theories

Number theory (or Peano Arithmetic) *PA* $\mathcal{L}$ contains the constant symbol 0, the 1-nary function symbol s, (for successor) and two 2-nary function symbol $+$ and $*$

1. $0 \neq s(x)$
2. $s(x) = s(y) \supset x = y$
3. $x + 0 = x$
4. $x + s(y) = s(x + y)$
5. $x * 0 = 0$
6. $x * s(y) = (x * y) + x$
7. the Induction axiom schema: $\phi(0) \wedge \forall x.(\phi(x) \supset \phi(s(x))) \supset \forall x.\phi(x)$, for every formula $\phi(x)$ with at least one free variable

K. Gödel 1931 It's false that $\mathcal{I} \models PA$ if and only if $\mathcal{I}$ is isomorphic to the standard models for natural numbers.