Mathematical Logic

$\mathcal{ALC}$ and more complex DLs

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Origins of Description Logics

Description Logics stem from early days knowledge representation formalisms (late '70s, early '80s):

- Semantic Networks: graph-based formalism, used to represent the meaning of sentences.
- Frame Systems: frames used to represent prototypical situations, antecedents of object-oriented formalisms.

Problems: **no clear semantics**, reasoning not well understood.

Description Logics (a.k.a. Concept Languages, Terminological Languages) developed starting in the mid '80s, with the aim of providing semantics and inference techniques to knowledge representation system

What are Description Logics today?

In the modern view, description logics are a family of logics that allow to speak about a domain composed of a set of generic (pointwise) objects, organized in classes, and related one another via various binary relations. Abstractly, description logics allows to predicate about labeled directed graphs

- vertexes represents real world objects
- vertexes's labels represents qualities of objects
- edges represents relations between (pairs of) objects
- vertexes' labels represents the types of relations between objects.

Every piece of world that can be abstractly represented in terms of a labeled directed graph is a good candidate for being formalized by a DL.

What are Description Logics about?

Exercise

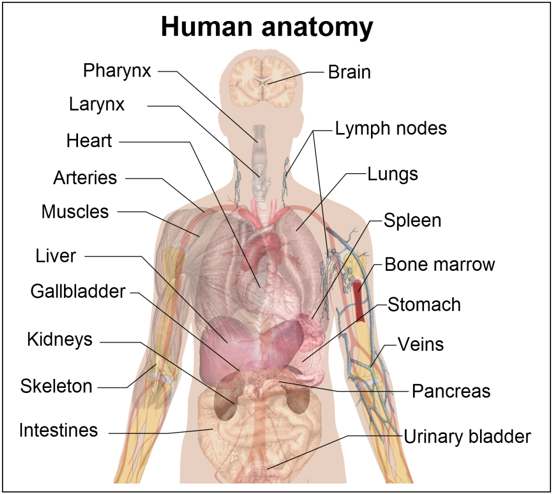
Represent Metro lines in Milan in a labelled directed graph

What are Description Logics about?

Exercise

Represent some aspects of Facebook as a labelled directed graph

What are Description Logics about?



Exercise

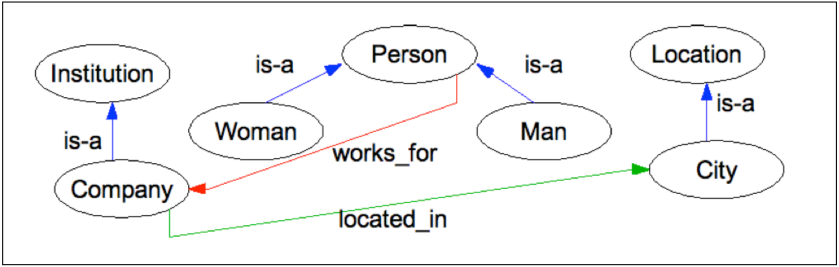
Represent some aspects of human anatomy as a labelled directed graph

What are Description Logics about?

Exercise

Represent some aspects of everyday life as a labelled directed graph

The everyday life example as a graph - intuition



- Family of logics designed for knowledge representation
- Allow to encode general knowledge (as above) as well as specific properties about objects (with individuals, e.g., *Mary*).

Ingredients of a Description Logic

A DL is characterized by:

1. A description language: how to form concepts and roles

$$\text{Human} \sqcap \text{Male} \sqcap \exists \text{hasChild}.\top \sqcap \forall \text{hasChild}.(\text{Doctor} \sqcup \text{Lawyer})$$

2. A mechanism to specify knowledge about concepts and roles (i.e., a TBox)

$$\mathcal{T} = \left\{ \begin{array}{l} \text{Father} \equiv \text{Human} \sqcap \text{Male} \sqcap \exists \text{hasChild}.\top \\ \text{HappyFather} \sqsubseteq \text{Father} \sqcap \forall \text{hasChild}.(\text{Doctor} \sqcup \text{Lawyer}) \\ \mathit{hasFather} \sqsubseteq \mathit{hasParent} \end{array} \right\}$$

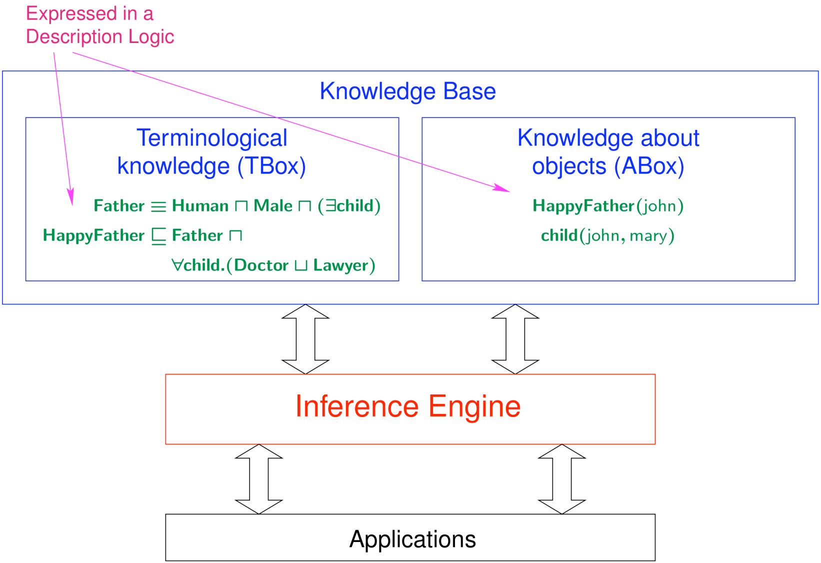
3. A mechanism to specify properties of objects (i.e., an ABox)

$$\mathcal{A} = \{\mathit{HappyFather}(\mathit{john}),\ \mathit{hasChild}(\mathit{john}, \mathit{mary})\}$$

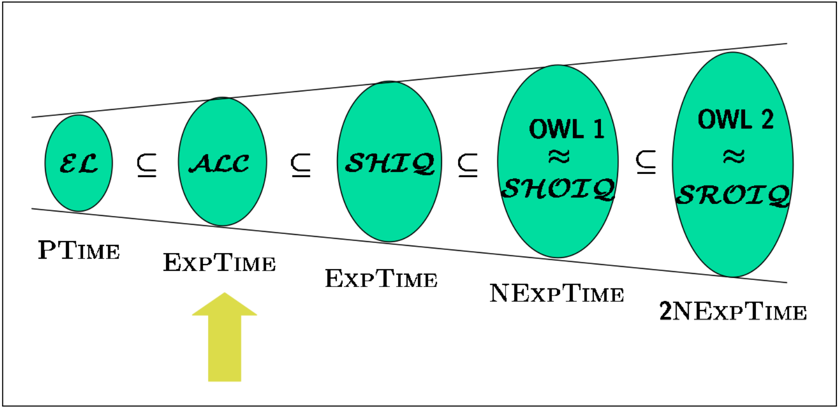
4. A set of inference services that allow to infer new properties on concepts, roles and objects, which are logical consequences of those explicitly asserted in the T-box and in the A-box

$$(\mathcal{T}, \mathcal{A}) \models \left\{ \begin{array}{l} \mathit{HappyFather} \sqsubseteq \exists \mathit{hasChild}.(\mathit{Doctor} \sqcup \mathit{Lawyer}) \\ \mathit{Doctor} \sqcup \mathit{Lawyer}(\mathit{mary}) \end{array} \right\}$$

Architecture of a Description Logic system



Many description logics



The description logics $\mathcal{ALC}$: Syntax

Alphabet

The alphabet Σ of $\mathcal{ALC}$ is composed of:

Σ_C:	Concept names	corresponding to node labels
Σ_R:	Role names	corresponding to arc labels
Σ_I:	Individual names	nodes identifiers

Grammar

Concept	$C := A \mid \neg C \mid C \sqcap C \mid \exists R.C$	$A \in \Sigma_C,\ R \in \Sigma_R$
Definition	$A \doteq C$	$A \in \Sigma_C$
Subsumption	$C \sqsubseteq C$	
Assertion	$C(a) \mid R(a, b)$	$a, b \in \Sigma_I,\ R \in \Sigma_R$

The description logics $\mathcal{ALC}$: Syntax

Abbreviations

$\top$	$A \sqcup \neg A$	for some $A \in \Sigma_C$
$\bot$	$\neg\top$	
$C \sqcup D$	$\neg(\neg C \sqcap \neg D)$	
$\forall R.C$	$\neg\exists R.(\neg C)$	
$C \equiv D$	$\{C \sqsubseteq D, D \sqsubseteq C\}$	

The metro example in $\mathcal{ALC}$

Exercise

Define Σ for speaking about the metro in Milan, and give examples of Concepts, Definitions, Subsumptions, and Assertions

Solution (Syntax)

- *Concept Names* (Σ_C):

Station	*the set of metro stations*
RedLineStation	*the set of metro stations on the red line*
ExchangeStation	*the set of metro stations where to change line*

- *Role Names* (Σ_R):

Next	*the relation between one station and its next stations*

- *Individual Names* (Σ_I):

Centrale	*the station called "Centrale"*
Gioia	*the station called "Gioia" . . .*

The metro example in $\mathcal{ALC}$ (Cont'd)

Solution (Concepts)

the set of stations which are on both the red and green line

$RedLineStation \sqcap GreenLineStation$

the set of exchange stations on the red line

$ExchangeStation \sqcap RedLineStation$

the set of stations which have a next station on the red line

$Station \sqcap \exists Next.RedLineStation$

The set of End stations

$Station \sqcap \forall Next.\bot$

The metro example in $\mathcal{ALC}$ (Cont'd)

Solution (Definitions)

$$
\begin{aligned}
\mathit{RGExchangeStation} &\doteq \mathit{RedLineStation} \sqcap \mathit{GreenLineStation} \\
\mathit{RYExchangeStation} &\doteq \mathit{RedLineStation} \sqcap \mathit{YellowLineStation} \\
\mathit{GYExchangeStation} &\doteq \mathit{GreenLineStation} \sqcap \mathit{YellowLineStation} \\
\mathit{ExchangeStation} &\doteq \mathit{RGExchangeStation} \sqcup \mathit{RYExchangeStation} \\
&\quad \sqcup \mathit{GYExchangeStation}
\end{aligned}
$$

The metro example in $\mathcal{ALC}$ (Cont'd)

Solution (Subsumptions)

A red line station is a station

$RedLineStation \sqsubseteq Station$

everything next to something is a station

$\top \sqsubseteq \forall Next.Station$

everything that has something next must be a station

$\exists Next.\top \sqsubseteq Station$

The metro example in $\mathcal{ALC}$ (Cont'd)

Solution (Assertions)

"Gioia" is a station of the green line

$GreenLineStation(Gioia)$

"Loreto" is an exchange station between the green and the red line

$RGExchangeStation(Loreto)$

"Lima" is the stop that follows "Loreto"

Next(Loreto,Lima)

"Duomo" is not the next stop of "Loreto"

$\neg Next(Loreto, Duomo)$

The description logics $\mathcal{ALC}$: Semantics

Definition

A DL interpretation $\mathcal{I}$ is pair $\langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ where:

- $\Delta^{\mathcal{I}}$ is a non empty set called interpretation domain
- $\cdot^{\mathcal{I}}$ is an interpretation function of the alphabet Σ such that
  - $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, every concept name is mapped into a subset of the interpretation domain
  - $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, every role name is mapped into a binary relation on the interpretation domain
  - $o^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ every individual is mapped into an element of the interpretation domain.

The description logics $\mathcal{ALC}$: Semantics

Interpretation of Complex concepts

$$
\begin{aligned}
(\neg C)^{\mathcal{I}} &= \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} \\
(C \sqcap D)^{\mathcal{I}} &= C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(\exists R.C)^{\mathcal{I}} &= \{d \in \Delta^{\mathcal{I}} \mid \text{exists } d', \langle d, d' \rangle \in R^{\mathcal{I}} \text{ and } d' \in C^{\mathcal{I}}\}
\end{aligned}
$$

Exercise

Provide the definition of the interpretations of the abbreviations:

$$
\begin{aligned}
(\top)^{\mathcal{I}} &= \ldots \\
(\bot)^{\mathcal{I}} &= \ldots \\
(C \sqcup D)^{\mathcal{I}} &= \ldots \\
(\forall R.C)^{\mathcal{I}} &= \ldots
\end{aligned}
$$

The description logics $\mathcal{ALC}$: Semantics

Satisfaction relation $\models$

$$
\begin{aligned}
\mathcal{I} \models A \doteq C \quad & \textit{iff} \quad A^{\mathcal{I}} = C^{\mathcal{I}} \\
\mathcal{I} \models C \sqsubseteq D \quad & \textit{iff} \quad C^{\mathcal{I}} \subseteq D^{\mathcal{I}} \\
\mathcal{I} \models C(a) \quad & \textit{iff} \quad a^{\mathcal{I}} \in C^{\mathcal{I}} \\
\mathcal{I} \models R(a,b) \quad & \textit{iff} \quad \langle a^{\mathcal{I}}, b^{\mathcal{I}} \rangle \in R^{\mathcal{I}}
\end{aligned}
$$

Satisfiability of a concept

A concept C is satisfiable if there is an interpretation $\mathcal{I}$, such that

$$C^{\mathcal{I}} \neq \emptyset$$

$\mathcal{ALC}$ knowledge base

Definition (Knowledge Base)

A knowledge base $\mathcal{K}$ is a pair $(\mathcal{T}, \mathcal{A})$, wehre

- $\mathcal{T}$, called the Terminological box (T-box), is a set of concept definition and subsumptions
- $\mathcal{A}$, called the Assertional box (A-box), is a set of assertions

Logical Consequence $\models$

A subsumption/assertion ϕ is a logical consequence of $\mathcal{T}$, $\mathcal{T} \models \phi$, if ϕ is satisfied by all interpretations that satisfies $\mathcal{T}$,

Satisfiability of a concept w.r.t, $\mathcal{T}$

A concept C is satisfiable w.r.t., $\mathcal{T}$ if there is an interpretation that satisfies $\mathcal{T}$ and such that

$$C^{\mathcal{I}} \neq \emptyset$$

$\mathcal{ALC}$ and First Order Logic

Remark

There is a strong relation between $\mathcal{ALC}$ and function free first order logics with unary and binary predicates

$\mathcal{ALC}$	$\longleftrightarrow$	First order logic
	$\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$	
concept name A	$\longleftrightarrow$	unary predicate $A(x)$
role name R	$\longleftrightarrow$	binary predicate $R(x, y)$
$\exists R.C$	$\longleftrightarrow$	$\exists y(R(x, y) \wedge C(y))$
$\neg C$	$\longleftrightarrow$	$\neg C(x)$
$C \sqcap D$	$\longleftrightarrow$	$C(x) \wedge D(x)$
$\mathcal{I} \models C(a)$	$\longleftrightarrow$	$\mathcal{I} \models C(a)$
$\mathcal{I} \models C \sqsubseteq D$	$\longleftrightarrow$	$\mathcal{I} \models \forall x(C(x) \rightarrow D(x))$

$\mathcal{ALC}$ and First Order Logics

Exercise

Define a transformation $\cdot^*$ from $\mathcal{ALC}$ concepts to first order formulas such that the following proposition is true

$$\models_{\mathcal{ALC}} \top \sqsubseteq C \quad \Rightarrow \quad \models_{FOL} C^*$$

Solution

$$
\begin{aligned}
ST^{x,y}(A) &= A(x) \\
ST^{x,y}(A \sqcap B) &= ST^{x,y}(A) \wedge ST^{x,y}(B) \\
ST^{x,y}(\neg A) &= \neg ST^{x,y}(A) \\
ST^{x,y}(\exists R.A) &= \exists y(R(x,y) \wedge ST^{y,x}(A))
\end{aligned}
$$

Exercise

Show that

1. $ST^{x,y}(C \sqcup D)$ is equivalent to $ST^{x,y}(C) \vee ST^{x,y}(D)$
2. $ST^{x,y}(\forall R.C)$ is equivalent to $\forall y(R(x,y) \rightarrow ST^{y,x}(C))$.

Relationship with First Order Logic – Exercise

Exercise

Translate the following $\mathcal{ALC}$ concepts in english and then in FOL

1. $Father \sqcap \forall.child.(Doctor \sqcup Manage)$
2. $\exists manages.(Company \sqcap \exists employs.Doctor)$
3. $Father \sqcap \forall child.(Doctor \sqcup \exists manages.(Company \sqcap \exists employs.Doctor))$

Solution

1. *fathers whose children are either doctors or managers*
   $Father(x) \wedge \forall y.(child(x,y) \rightarrow (Doctor(y) \vee Manager(y)))$
2. *those who manages a company that employs at least one doctor*
   $\exists y.(manages(x,y) \wedge (Company(y) \wedge \exists x.(employs(y,x) \wedge Doctor(x)))$
3. *fathers whose children are either doctors or managers of companies that employ some doctor.*
   $Father(x) \wedge \forall y.(child(x,y) \rightarrow (Doctor(y) \vee \exists x.(manages(y,x) \wedge (Company(x) \wedge \exists y.(employs(x,y) \wedge Doctor(y))))))$

$\mathcal{ALC}$ and First Order Logics

Two Variables First Order Logics (FO^2)

A k-variable first order logic, FO^k is a logic defined on a First Order Language without functional symbols and with k individual variables. FO^2 is the first order logic with at most two variables

Theorem

The satisfiability problem for FO^2 is NEXPTIME *complete. (Erich Grädel, Phokion G. Kolaitis, Moshe Y. Vardi, On the Decision Problem for Two-Variable First-Order Logic, The Bulletin of Symbolic Logic, Volume 3, Number 1, March 1997,* `http://www.math.ucla.edu/ asl/bsl/0301/0301-003.ps` *)*

ALC is a fragment of FO^2. However FOL with 2 variables is more expressive than ALC (left for advanced courses in Logic for KR).

Numeric constraints

- Functionality restrictions $\mathcal{ALCF}$: allow one to impose that a relation is a function:
  - global functionality: $\top \sqsubseteq (\leq 1\, R)$ (equivalent to $(\textbf{funct}\ R)$)
    Example: $\top \sqsubseteq (\leq 1\, \text{hasFather})$
  - local functionality: $A \sqsubseteq (\leq 1\, R)$
    Example: $\text{Person} \sqsubseteq (\leq 1\, \text{hasFather})$

- Number restrictions $\mathcal{ALCN}$: $(\leq n\, R)$ and $(\geq n\, R)$
  Example: $\text{Person} \sqsubseteq (\leq 2\, \text{hasParent})$

- Qualified Number restrictions $\mathcal{ALCQ}$: $(\leq n\, R.\, C)$ and $(\geq n\, R.\, C)$
  Example:
  $$\begin{aligned}\text{FootballTeam} \sqsubseteq\ & (\geq 1\, \text{hasPlayer}.\, \text{Golly}) \sqcap \\ & (\leq 1\, \text{hasPlayer}.\, \text{Golly}) \sqcap \\ & (\geq 2\, \text{hasPlayer}.\, \text{Defensor}) \sqcap \\ & (\leq 4\, \text{hasPlayer}.\, \text{Defensor})\end{aligned}$$

Role constructs

- Inverse roles $\mathcal{ALCI}$: R^-, interpreted as $(R^-)^\mathcal{I} = \{(y, x) \mid (x, y) \in R^\mathcal{I}\}$
  Example:
  we can refer to the parent, by using the hasChild role, e.g., $\exists$hasChild$^-$.Doctor.

- Transitive roles: (**trans** R), stating that the relation $R^\mathcal{I}$ is transitive, i.e., $\{(x, y), (y, z)\} \subseteq R^\mathcal{I} \rightarrow (x, z) \in R^\mathcal{I}$
  Example: (**trans** hasAncestor)

- Subsumption between roles: $R_1 \sqsubseteq R_2$, used to state that a relation is contained in another relation.
  Example: hasMother $\sqsubseteq$ hasParent

$\mathcal{ALC}$ language - exercises

Exercise

Let Man, Woman, Male, Female, and Human be concept names, and let has-child, is-brother-of, is-sister-of, and is-married-to be role names.
Try to construct a T-box that contains definitions for

Mother	Grandfather	Niece
Father	Aunt	Nephew
Grandmother	Ancle	Mother-of-at-least-one-male

$\mathcal{ALC}$ Language - exercises

Exercise

Express the following sentences in terms of the description logic $\mathcal{ALC}$

1. All employees are humans.
   $employee \sqsubseteq human$
2. A mother is a female who has a child.
   $mother \equiv female \sqcap \exists hasChild.\top$
3. A parent is a mother or a father.
   $parent \equiv mather \sqcup father$
4. A grandmother is a mother who has a child who is a parent.
   $grandmother \equiv mother \sqcap \exists hasChild.parent$
5. Only humans have children that are humans.
   $\exists hasChild.human \sqsubseteq human$

$\mathcal{ALC} \rightarrow FOL$ - exercises

Exercise

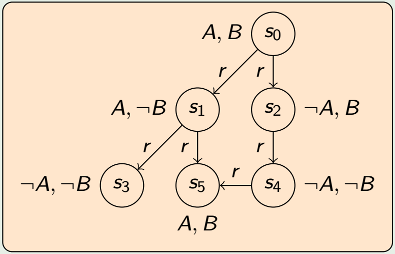
Translate the following inclusion axioms in the language of First order logic

$Female \sqsubseteq Human$	females are humans
$Child \sqsubseteq Human$	children are humans
$StudiesAtUni \sqsubseteq Human$	university students are humans
$SuccessfullMan \equiv Man \sqcap$	a successful man is a man who
$InBusiness \sqcap \exists married.Lawyer \sqcap$	is in business, has married a lawyer
$\exists hasChild.(StudiesAtUni)$	and has a child who is a student
$\neg Female(Pedro)$	Pedro is not a female
$InBusiness(Pedro)$	Pedro is in business
$Lawyer(Mary)$	Mary is a lawyer
$married(Pedro, Mary)$	pedro is married with Mary
$child(Pedro, John)$	John is the child of Pedre

Satisfaction - exercise

Exercise

Let $\mathcal{I}$ be the following $\mathcal{ALC}$ interpretation on the domain $\Delta^{\mathcal{I}} = \{s_0, s_1, \ldots, s_5\}$. Calculate the interpretation of the following concepts:



$$\top^{\mathcal{I}} = \{s_0, s_1, \ldots, s_5\}$$

$$\bot^{\mathcal{I}} = \emptyset$$

$$A^{\mathcal{I}} = \{s_0, s_1, s_5\}$$

$$B^{\mathcal{I}} = \{s_0, s_2, s_5\}$$

$$(A \sqcap B)^{\mathcal{I}} = \{s_0, s_5\}$$

$$(A \sqcup B)^{\mathcal{I}} = (\{s_0, s_1, s_2, s_5\})$$

$$(\neg A)^{\mathcal{I}} = \{s_2, s_3, s_4\}$$

$$(\exists r.A)^{\mathcal{I}} = \{s_0, s_1, s_4\}$$

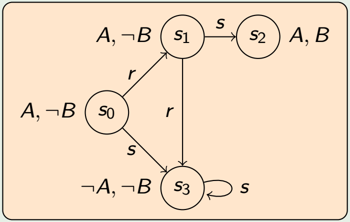
$$(\forall r.\neg B)^{\mathcal{I}} = \{s_3, s_2\}$$

$$(\forall r.(A \sqcup B))^{\mathcal{I}} = \{s_0, s_3, s_4\}$$

Satisfaction - exercise

Exercise

Let $\mathcal{I}$ be the following $\mathcal{ALC}$ interpretation on the domain $\Delta^{\mathcal{I}} = \{s_0, s_1, \ldots, s_5\}$. Calculate the interpretation of the following concepts:



$$(A \sqcup B)^{\mathcal{I}} = \{s_0, s_1, s_2\}$$

$$(\exists s.\neg A)^{\mathcal{I}} = \{s_0, s_1, s_3\}$$

$$(\forall s.A)^{\mathcal{I}} = \{s_2\}$$

$$(\exists s.\exists s.\exists s.\exists s.A)^{\mathcal{I}} = \emptyset$$

$$(\neg \exists r.(\neg A \sqcup \neg B))^{\mathcal{I}} = \{s_1, s_2\}$$

$$(\exists s.(A \sqcup \forall s.\neg B) \sqcup \neg \forall r.\exists r.(A \sqcup \neg A))^{\mathcal{I}} = \{s_0, s_1, s_3\}$$

$\mathcal{ALC}$ satisfaction - exercises

Exercise

Consider an ALC-signature with atomic concepts $\Sigma_c = \{A, B\}$ and role names $\Sigma_R = \{R, S\}$ and an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ given by

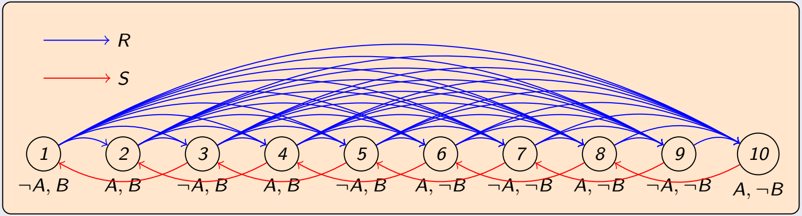
- $\Delta^{\mathcal{I}} = \{1, 2, 3, \dots, 10\}$
- $A^{\mathcal{I}} = \{n \in \Delta^{\mathcal{I}} | n \text{ is even}\}$
- $B^{\mathcal{I}} = \{n \in \Delta^{\mathcal{I}} | n \leq 5\}$
- $R^{\mathcal{I}} = \{(x, y) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} | x < y\}$
- $S^{\mathcal{I}} = \{(x, y) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} | x - y = 2\}$

Compute the interpretation $C^{\mathcal{I}}$ for each of the concepts C below

1. $C = \exists S.\forall R.\bot$
2. $C = \forall S.\exists R.B$
3. $C = \neg\exists S.(B \sqcap \forall R.A)$

Solution

$\mathcal{I}$ can be graphically represented by the following graph:



1. $(\exists S.\forall R.\bot)^{\mathcal{I}}$ = *the set of nodes that have an outgoing S-arc that reaches a node with no outgoing R-arcs. (notice that $\forall R.\bot$ is satisfied by the nodes that do not have outgoing R-arcs. I.e.,* $\emptyset$
2. $(\forall S.\exists R.B)^{\mathcal{I}}$ = *the set of nodes such that every outgoing S-arc reaches a node for which there is an outgoing R arch that reaches a node ≤ 5. I.e.,* $\{1, 2, 3, 4, 5, 6\}$
3. $(\neg\exists S.(B \sqcap \forall R.A))^{\mathcal{I}}$ = *the set of nodes for which there is no outgoing S-arc reaching a node ≤ 5 and such that all its outgoing R-arcs reaches an even number. I.e.,* $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

$\mathcal{ALC}$ general properties - exercises

Exercise

Show that $\models C \sqsubseteq D$ implies $\models \exists R.C \sqsubseteq \exists R.D$

Solution

We have to prove that for all $\mathcal{I}$, $(\exists R.C)^{\mathcal{I}} \subseteq (\exists R.C)^{\mathcal{I}}$ under the hypothesis that for all $\mathcal{I}$, $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.

- *Let $x \in (\exists R.C)^{\mathcal{I}}$, we want to show that x is also in $(\exists R.D)^{\mathcal{I}}$.*
- *If $x \in (\exists R.C)^{\mathcal{I}}$, then by the interpretation of $\exists R$ there must be an y with $(x, y) \in R^{\mathcal{I}}$ such that $y \in C^{\mathcal{I}}$.*
- *By the hypothesis that $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ for all $\mathcal{I}$, we have that $y \in D^{\mathcal{I}}$.*
- *The fact that $(x, y) \in R^{\mathcal{I}}$ and $y \in D^{\mathcal{I}}$ implies that $x \in (\exists R.D)^{\mathcal{I}}$.*

$\mathcal{ALC}$ (un)satisfiability and validity - exercises

Exercise

For each of the following formula say if it is valid, satisfiable or unsatisfiable. If it is not valid provide a model that falsify it.

$$\forall R(A \sqcap B) \equiv \forall RA \sqcap \forall RB$$
$$\forall R(A \sqcup B) \equiv \forall RA \sqcup \forall RB$$
$$\exists R(A \sqcap B) \equiv \exists RA \sqcap \exists RB$$
$$\exists R(A \sqcup B) \equiv \exists RA \sqcup \exists RB$$

Solution

$\forall R(A \sqcap B) \equiv \forall RA \sqcup \forall RB$ is valid and we can prove that $(\forall R(A \sqcap B))^{\mathcal{I}} = (\forall R.A \sqcap \forall R.B)^{\mathcal{I}}$ for all interpretations $\mathcal{I}$.

$$\begin{aligned}
(\forall R(A \sqcap B))^{\mathcal{I}} &= \{(x,y) \in R^{\mathcal{I}} \mid y \in (A \sqcap B)^{\mathcal{I}}\} \\
&= \{(x,y) \in R^{\mathcal{I}} \mid y \in A^{\mathcal{I}} \cap B^{\mathcal{I}}\} \\
&= \{(x,y) \in R^{\mathcal{I}} \mid y \in A^{\mathcal{I}}\} \cap \{(x,y) \in R^{\mathcal{I}} \mid y \in B^{\mathcal{I}}\} \\
&= (\forall R.A)^{\mathcal{I}} \cap (\forall R.B)^{\mathcal{I}} \\
&= (\forall R.A \sqcap \forall R.B)^{\mathcal{I}}
\end{aligned}$$

$\mathcal{ALC}$ (un)satisfiability and validity - exercises

Exercise

For each of the following formula say if it is valid, satisfiable or unsatisfiable. If it is not valid provide a model that falsify it.

$$\forall R(A \sqcap B) \equiv \forall RA \sqcap \forall RB$$

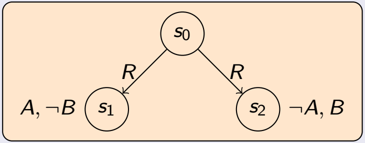
$$\forall R(A \sqcup B) \equiv \forall RA \sqcup \forall RB$$

$$\exists R(A \sqcap B) \equiv \exists RA \sqcap \exists RB$$

$$\exists R(A \sqcup B) \equiv \exists RA \sqcup \exists RB$$

Solution

$\forall R(A \sqcup B) \equiv \forall RA \sqcup \forall RB$ is not valid. The following model is such that $(\forall R(A \sqcup B))^{\mathcal{I}} \neq (\forall RA \sqcup \forall RB)^{\mathcal{I}}$



- $s_0 \in (\forall R(A \sqcup B))^{\mathcal{I}}$ *but*
- $s_0 \notin (\forall RA)$ *and*
- $s_0 \notin (\forall RB)^{\mathcal{I}}$

However notice that the containment: $\forall R.A \sqcup \forall R.B \sqsubseteq \forall R.(A \sqcup B)$ is valid

$\mathcal{ALC}$ (un)satisfiability and validity - exercises

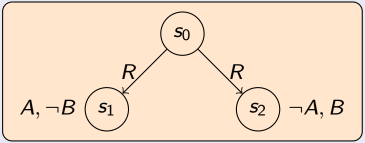
Exercise

For each of the following formula say if it is valid, satisfiable or unsatisfiable. If it is not valid provide a model that falsify it.

$$\forall R(A \sqcap B) \equiv \forall RA \sqcap \forall RB$$
$$\forall R(A \sqcup B) \equiv \forall RA \sqcup \forall RB$$
$$\exists R(A \sqcap B) \equiv \exists RA \sqcap \exists RB$$
$$\exists R(A \sqcup B) \equiv \exists RA \sqcup \exists RB$$

Solution

$\exists R(A \sqcap B) \equiv \exists RA \sqcap \exists RB$ is not valid. The following model is such that $(\exists R(A \sqcap B))^{\mathcal{I}} \neq (\exists RA \sqcap \exists RB)^{\mathcal{I}}$



- *$s_0 \in (\exists RA)^{\mathcal{I}}$ and*
- *$s_0 \in (\exists RB)^{\mathcal{I}}$ but*
- *$s_0 \notin (\exists R(A \sqcap B))^{\mathcal{I}}$*

However notice that the containment: $\exists R(A \sqcap B) \sqsubseteq \exists RA \sqcap \exists RB$ is valid

$\mathcal{ALC}$ (un)satisfiability and validity - exercises

Exercise

For each of the following formula say if it is valid, satisfiable or unsatisfiable. If it is not valid provide a model that falsify it.

$$\forall R(A \sqcap B) \equiv \forall RA \sqcap \forall RB$$

$$\forall R(A \sqcup B) \equiv \forall RA \sqcup \forall RB$$

$$\exists R(A \sqcap B) \equiv \exists RA \sqcap \exists RB$$

$$\exists R(A \sqcup B) \equiv \exists RA \sqcup \exists RB$$

Solution

$\exists R(A \sqcup B) \equiv \exists RA \sqcup \exists RB$ *is valid. We can provide a proof similar to the case of* $\forall R.(A \sqcap B) \equiv \forall R.A \sqcap \forall R.B$, *but in the following we provide an alternative proof, which is based on other equivalences:*

$$\begin{aligned}
\exists R(A \sqcup B) &\equiv \neg\forall R(\neg(A \sqcup B)) \\
&\equiv \neg\forall R.(\neg A \sqcap \neg B) \\
&\equiv \neg(\forall R.(\neg A) \sqcap \forall R.(\neg B)) \\
&\equiv \neg(\forall R.(\neg A) \sqcup \neg\forall R.(\neg B) \\
&\equiv \exists R.A \sqcup \exists R.B
\end{aligned}$$

Exercise

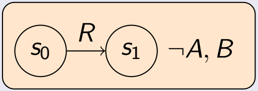
For each of the following concept say if it is valid, satisfiable or unsatisfiable. If it is valid, or unsatisfiable, provide a proof. If it is satisfiable (and not valid) then exhibit a model that interprets the concept in a non-empty set

1. $\neg(\forall R.A \sqcup \exists R.(\neg A \sqcap \neg B))$
2. $\exists R.(\forall S.C) \sqcap \forall R.(\exists S.\neg C)$
3. $(\exists S.C \sqcap \exists S.D) \sqcap \forall S.(\neg C \sqcup \neg D)$
4. $\exists S.(C \sqcap D) \sqcap (\forall S.\neg C \sqcup \exists S.\neg D)$
5. $C \sqcap \exists R.A \sqcap \exists R.B \sqcap \neg\exists R.(A \sqcap B)$

$\mathcal{ALC}$ (un)satisfiability and validity - exercises

Solution

1. $\neg(\forall R.A \sqcup \exists R.(\neg A \sqcap \neg B))$ *Satisfiable*



$$s_0 \in (\neg(\forall R.A \sqcup \exists R.(\neg A \sqcap \neg B))^{\mathcal{I}}$$
$$s_1 \notin (\neg(\forall R.A \sqcup \exists R.(\neg A \sqcap \neg B))^{\mathcal{I}}$$

2. $\exists R.(\forall S.C) \sqcap \forall R.(\exists S.\neg C)$ *unsatisfiable, since* $\exists R.\forall S.C \equiv \neg\forall R.\neg\forall S.C \equiv \neg\forall R.\exists S.\neg C$. *This implies that* $\exists R.(\forall S.C) \sqcap \forall R.(\exists S.\neg C)$ *is equivalent to* $\neg(\forall R.\exists S.\neg C) \sqcap (\forall R.\exists S.\neg C)$, *which is a concept of the form* $\neg B \sqcap B$ *which is always unsatisfiable.*
3. $(\exists S.C \sqcap \exists S.D) \sqcap \forall S.(\neg C \sqcup \neg D)$ *satisfiable*
4. $\exists S.(C \sqcap D) \sqcap (\forall S.\neg C \sqcup \exists S.\neg D)$ *unsatisfiable*
5. $C \sqcap \exists R.A \sqcap \exists R.B \sqcap \neg\exists R.(A \sqcap B)$ *satisfiable*

Exercise

Check if the following subsumption is valid

$$\neg\forall R.A \sqcap \forall R((\forall R.B) \sqcup A) \sqsubseteq \forall R.\neg(\exists R.A) \sqcap \exists R.(\exists R.B)$$