Propositional Logic

Exercise 1. [**4 marks**] Show that the formulas $p \to (q \to r)$ and $(p \to q) \to r$ are not equivalent by finding an interpretation in which they have different truth values.

Solution. The two formulas are equivalent to $\neg p \vee \neg q \vee r$ and $(p \wedge \neg q) \vee r$, respectively. It is easy to observe that the interpretation which assigns false to p, q and r makes true the first formula and false the second formula.

Exercise 2. [**4 marks**] Verify by using the DPLL algorithm the validity of the formula $(R \wedge (P \vee \neg Q)) \to P$

Solution. To prove the validity of the formula we need to show that the negation of the formula is unsatisfiable. The negation of the formula is: $R \wedge (P \vee \neg Q) \wedge \neg P$
Let us assign a truth value to the unit clauses: v(R) = T, v(P) = F.
After the propagation we obtain the simplified formula $\neg Q$.
Since Q appears pure we can conclude that the formula is satisfiable and therefore the original formula is not valid.

Exercise 3. [**3 marks**] Prove the soundness of the $\wedge I$ rule of Natural Deduction.

$$\frac{\phi \quad \psi}{\phi \wedge \psi} \wedge I$$

Solution. Assume that $\Gamma \vdash_{ND} \phi \wedge \psi$ and the last rule used is $\wedge I$, then from the shape of the rule we know that there are two deductions of ϕ and ψ from two sets Γ_1 and Γ_2 with $\Gamma_1 \subseteq \Gamma$ and $\Gamma_2 \subseteq \Gamma$. In symbols this corresponds to

$$\Gamma_1 \vdash_{ND} \phi \tag{1}$$
$$\Gamma_2 \vdash_{ND} \psi \tag{2}$$

From the inductive hypothesis, (1) and (2) imply that

$$\Gamma_1 \models \phi \tag{3}$$
$$\Gamma_2 \models \psi \tag{4}$$

and because of the monotonicity of logical consequence in propositional logic we have that

$$\Gamma \models \phi \tag{5}$$
$$\Gamma \models \psi \tag{6}$$

Now we can prove that $\Gamma \models \phi \wedge \psi$. In fact, let $\mathcal{I}$ be an interpretation that satisfies Γ ($\mathcal{I} \models \Gamma$). From (5) and (6) we know that $\mathcal{I}$ satisfies both ϕ and ψ ($\mathcal{I} \models \phi$ and $\mathcal{I} \models \psi$). Therefore, from the definition of satisfiability of $\wedge$ we have that $\mathcal{I}$ satisfies $\phi \wedge \psi$ ($\mathcal{I} \models \phi \wedge \psi$).

First order logics

Exercise 4. [3 marks] Prove by natural deduction that the following formula is valid:

$$\forall xy(P(x) \to P(y)) \to (\exists x P(x) \to \forall y P(y))$$

Solution.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall xy(P(x) \to P(y))]^{(1)}}{\forall y(P(c) \to P(y))}\,\forall \mathrm{E}}{P(c) \to P(d)}\,\forall \mathrm{E}}{P(c) \to \forall y P(y)}\,\forall \mathrm{I}}{\exists x P(x) \to \forall y P(y)}\,\exists \mathrm{I}}{\forall xy(P(x) \to P(y)) \to (\exists x P(x) \to \forall y P(y))}\,\supset \mathrm{I}^{(1)}$$

Exercise 5. [3 marks] Represent in FOL the following natural language sentences :

1. There are at least two mountains located in England
2. There are exactly two coins in the box
3. No mountain is higher than itself

Solution. The three sentences can be represented as follows:

1. $\exists xy(Mountain(x) \wedge Mountain(y) \wedge LocatedIn(x, England) \wedge LocatedIn(y, England) \wedge \neg(x = y))$
2. $\exists xy(Coin(x) \wedge InBox(x) \wedge Coin(y) \wedge InBox(y) \wedge \neg(x = y) \wedge \forall z(Coin(z) \wedge InBox(z)) \to (x = z \vee y = z)$
3. $\forall x Mountain(x) \to \neg Higher(x, x)$

Exercise 6. [5 marks] 5 marks. Let L be a first order language that contains only a binary relation R(x,y) and a constant r. Write a set of axioms A, so that for every interpretation I $\models$ A if and only if I forms a tree with root r and depth at most equal to 3.

Solution. A possible set of axioms A is as follows:

1. $\forall y \neg R(y, r)$ *(r is the root)*

2. $\forall x(r = x \vee R(r, z) \vee \exists y(R(r, y) \wedge R(y, x)) \vee \exists yz(R(r, z) \wedge R(z, y) \wedge R(y, x))$ *(every element different form the root can be reached in at most 2 steps from the root)*

3. $\forall xyz(R(x, y) \wedge R(z, y) \rightarrow z = x)$ *(there is only one father)*

4. $\forall xyR(x, y) \rightarrow \neg R(y, x)$

5. $\forall xyz(R(x, y) \wedge (R(y, z) \rightarrow \neg R(z, x))$

6. $\forall xyz(R(x, y) \wedge R(y, z) \wedge R(z, w) \rightarrow \forall v \neg R(w, v))$

Description Logics

Exercise 7. [**4 marks**] Given the following TBox T:
$\{Undergraduate \sqsubseteq \neg Teach, PhD \equiv Master \sqcap Research, Bachelor \equiv Student \sqcap Undergraduate, Assistant \equiv PhD \sqcap Teach, Master \equiv Student \sqcap \neg Undergraduate\}$ prove by concept expansion that $T \models Assistant \sqsubseteq Student$.

Solution. The following chain of expansions proves it:
$Assistant \equiv PhD \sqcap Teach$
$PhD \sqcap Teach \equiv (Master \sqcap Research) \sqcap Teach$
$Master \sqcap Research \sqcap Teach \equiv (Student \sqcap \neg Undergraduate) \sqcap Research \sqcap Teach$
$(Student \sqcap \neg Undergraduate) \sqcap Research \sqcap Teach \sqsubseteq Student$
$Student \sqsubseteq Student, andthereforeAssistant \sqsubseteq Student$

Exercise 8. [**4 marks**] Given the TBox T = $\{A \sqsubseteq B \sqcap \neg C, C \sqsubseteq E \sqcap B\}$ and the ABox A = $\{A(a), B(b)\}$,

1. Normalize T

2. Expand the normalized T

3. Expand A

4. Provide the instance retrieval of E

Solution. For the four points above:

1. $A \equiv B \sqcap \neg C \sqcap A1, C \equiv E \sqcap B \sqcap C1$

2. $A \equiv B \sqcap \neg(E \sqcap B \sqcap C1) \sqcap A1, C \equiv E \sqcap B \sqcap C1$ (notice that this is equivalent to $A \equiv B \sqcap (\neg E \sqcup B \sqcup \neg C1) \sqcap A1, C \equiv E \sqcap B \sqcap C1$)

3. B(b), A(a), B(a), A(a) and one or more of the following $\neg E(a), \neg B(a), \neg C(a)$

4. It is empty

Exercise 9. [3 marks] Given the TBox T = $\{A \equiv \neg B \sqcap \neg C, B \equiv D \sqcap C\}$ and the ABox A = $\{\neg B(a), C(a), A(a)\}$, check if A is consistent with T via expansion of A.

Solution. It is not. In fact, by expanding A(a) we obtain $\neg B(a)$ and $\neg C(a)$, thus having both C(a) and $\neg C(a)$ in the ABox.