Logics for Data and Knowledge Representation

6. DLs more expressive than $\mathcal{ALC}$

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Extensions of $\mathcal{ALC}$

Number restrictions $\mathcal{ALCN}$ $(\leq n)R$ $[(\geq n)R]$

$$Persons \sqsubseteq (\leq 1)\text{is\_merried\_with}$$

Number restriction allows to impose that a relation is a function

Qualified Number restrictions $\mathcal{ALCQ}$ $(\leq n)R.C$ $[(\geq n)R.C]$

$$\begin{aligned}\text{football\_team} \quad \sqsubseteq \quad & (\geq 1)\text{has\_player.Golly} \sqcap \\ & (\leq 2)\text{has\_player.Golly} \sqcap \\ & (\geq 2)\text{has\_player.Defensor} \sqcap \\ & (\geq 4)\text{has\_player.Defensor} \sqcap \\ & \ldots\end{aligned}$$

Extensions of $\mathcal{ALC}$

Inverse roles $\mathcal{ALCI}$ R^-. make it possible to use the inverse of a role. For example, we can specify has_Parent as the inverse of has_Child,

$$\text{has\_Parent} \equiv \text{has\_Child}^-$$

meaning that $\text{hasParent}^\mathcal{I} = \{(y, x) \mid (x, y) \in \textit{has\_ChildI}^\mathcal{I}\}$

Transitive roles tr(R) used to state that a given relation is transitive

$$Tr(hasAncestor)$$

meaning that $(x, y), (y, z) \in \text{hasAncestor}^\mathcal{I} \rightarrow (x, z) \in \text{hasAncestor}^\mathcal{I}$

Subsumptions between roles $R \sqsubseteq S$ used to state that a relation is contained in another relation.

$$hasMother \sqsubseteq hasParent$$

Modeling with Inverse role

Exercise

Try to model the following facts in $\mathcal{ALCI}$. (notice that not all the statements are modellable in $\mathcal{ALCI}$)

1. Lonely people do not have friends and are not friends of anybody
2. An intermediate stop is a stop which has a predecessor stop and a successor stop
3. A person is a child of his father

Modeling with Inverse role

Solution

1. *Lonely people do not have friends and are not friends of anybody*
2. *An intermediate stop is a stop which has a predecessor stop and a successor stop*
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1. *Lonely people do not have friends and are not friends of anybody*

   $$lonely\_person \equiv person \sqcap \neg\exists has\_friend^{-}.\top \sqcap \neg\exists has\_friend.\top$$

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2. *An intermediate stop is a stop which has a predecessor stop and a successor stop*

$$Intermediate\_stop \equiv Stop \sqcap \exists next.Stop \sqcap \exists next^{-}.Stop$$

3. *A person is a child of his father*

non modellable

$$Person \sqsubseteq \forall has\_father(\forall has\_father^{-}.Person)$$

is not enough

Expressiveness of Inverse role

Exercise

Prove that the inverse role primitive constitutes an effective extension of the expressivity of $\mathcal{ALC}$, i.e., show that that $\mathcal{ALC}$ is strictly less expressive than $\mathcal{ALCI}$.

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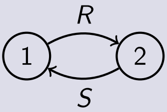

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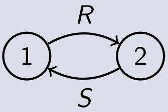

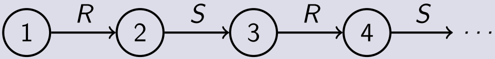

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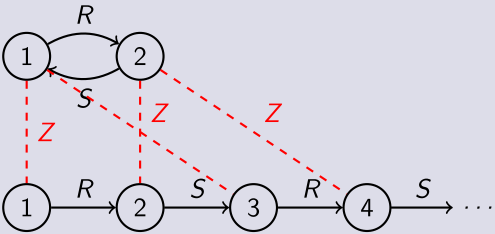

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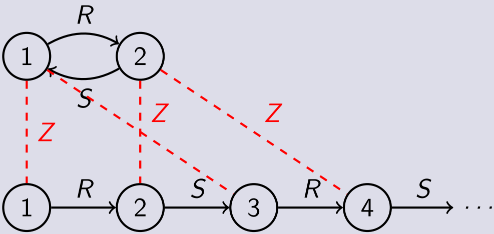

$\models \exists R.\top \sqsubseteq \exists S^{-}.\top$

$\not\models \exists R.\top \sqsubseteq \exists S^{-}.\top$

Properties of $\mathcal{ALCI}$ models

Theorem (Tree model property)

If C is satisfiable w.r.t. a T-box $\mathcal{T}$, then it is satisfiable w.r.t. $\mathcal{T}$ by a tree-shaped model with root an element of C.

Proof.

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1. extend the notion of bisimulation for $\mathcal{ALCI}$

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Proof.

1. extend the notion of bisimulation for $\mathcal{ALCI}$
2. show that if $(\mathcal{I}, d) \sim_{\mathcal{ALCI}} (\mathcal{J}, e)$, then $d \in C^{\mathcal{I}}$ iff $e \in C^{\mathcal{J}}$ for any $\mathcal{ALCI}$ concept C

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2. show that if $(\mathcal{I}, d) \sim_{\mathcal{ALCI}} (\mathcal{J}, e)$, then $d \in C^{\mathcal{I}}$ iff $e \in C^{\mathcal{J}}$ for any $\mathcal{ALCI}$ concept C
3. For a non tree-shaped model $\mathcal{I}$ and any element d, generate a tree-shaped model $\mathcal{J}$ rooted at e and show that $(\mathcal{I}, d) \sim_{\mathcal{ALCI}} (\mathcal{J}, e)$.

□

Bisimulation for $\mathcal{ALCI}$]

Definition ($\mathcal{ALCI}$-Bisimulation)

A $\mathcal{ALCI}$-bisimulation ρ between two $\mathcal{ALCI}$ interpretations $\mathcal{I}$ and $\mathcal{J}$ is a bisimulation ρ, that satisfies the following additional condition when $d \rho e$:

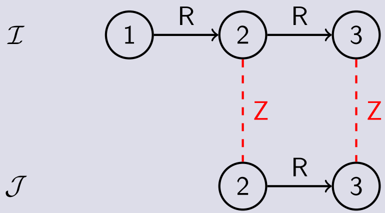
Inverse relation equivalence

- for all d' such that $(d', d) \in R^{\mathcal{I}}$, there is an $e' \in \Delta^{\mathcal{J}}$ such that $(e', e) \in R^{\mathcal{J}}$ and $d' \rho e'$.
- Same property in the opposite direction

$(\mathcal{I}, d) \sim_{\mathcal{ALCI}} (\mathcal{J}, e)$ means that there is a $\mathcal{ALCI}$-bisimulation ρ between $\mathcal{I}$ and $\mathcal{J}$ such that $e \rho e$.

$\mathcal{ALCI}$-bisimulation

Example of bisimulation which is not a $\mathcal{ALCI}$-bisimulation, and how should be



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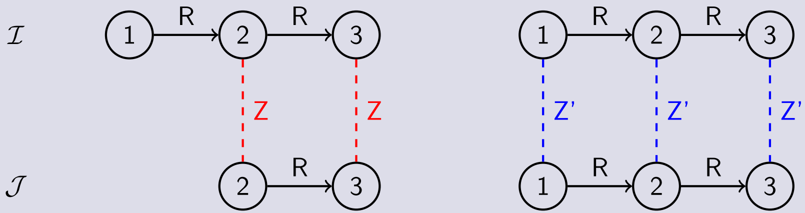

$(\mathcal{I}, 2) \sim (\mathcal{J}, 2)$ but not $(\mathcal{I}, 1) \sim_{\mathcal{ALCI}} (\mathcal{J}, 1)$

Invariance under $\mathcal{ALCI}$-bisimulation

Theorem

If $(\mathcal{I}, d) \sim_{\mathcal{ALCI}} (\mathcal{J}, e)$, then $d \in C^{\mathcal{I}}$ iff $e \in C^{\mathcal{J}}$ for any $\mathcal{ALCI}$ concept C

Proof.

by induction on the complexity of C. All the cases as in $\mathcal{ALC}$, in addition we have the following step cases

- **if C is $\exists R^{-}.C$**

$$
\begin{array}{lll}
\mathcal{I}, d \models \exists R^{-}.C & \textit{iff} & \mathcal{I}, d' \models C \text{ for some } d' \text{ with } (d', d) \in R^{\mathcal{I}} \\
& \textit{iff} & \mathcal{J}, e' \models C \text{ for some } e' \text{ with } (e', e) \in R^{J} \\
& & \text{and } (\mathcal{I}, d') \sim_{\mathcal{ALCI}} (\mathcal{J}, e') \\
& \textit{iff} & \mathcal{J}, e \models \exists R^{-}.C
\end{array}
$$

□

Transformation in tree-shaped $\mathcal{ALCI}$ models

Theorem

If $\mathcal{I}$ is a non tree-shaped model, and d any element of $\mathcal{I}$, then there is a model $\mathcal{J}$ which is tree-shaped such that $(\mathcal{I}, d) \sim_{\mathcal{ALCI}} (\mathcal{J}, d)$.

Proof.

We define $\mathcal{J}$ as follows:

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Proof.

We define $\mathcal{J}$ as follows:

- $\Delta_{\mathcal{J}}$ is the set of paths $\pi = (d_1, d_2, \ldots, d_n)$ such that $d_1 = d$, and $(d_i, d_{i+1}) \in R_i$ or $(d_{i+1}, d_i) \in R_i^{\mathcal{I}}$ for $(1 \leq i \leq n - 1)$.
- $A^{\mathcal{J}} = \{\pi d_n | d_n \in A^{\mathcal{I}}\}$
- $$R^{\mathcal{J}} = \begin{array}{l} \{(\pi d_n \ , \ \pi d_n d_{n+1}) | (d_n, d_{n+1}) \in R^{\mathcal{I}}\} \cup \\ \{(\pi d_n d_{n+1} \ , \ \pi d_n) | (d_{n+1}, d_n) \in R^{\mathcal{I}}\} \end{array}$$

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It's easy to show that $\mathcal{J}$ is a tree-shaped model rooted at d

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It's easy to show that $\mathcal{J}$ is a tree-shaped model rooted at d
The $\mathcal{ALCI}$ bisimulation ρ between $\mathcal{I}$ and $\mathcal{J}$ is defined as
$(d_i), \pi d_i) \in \rho$ □

Number restriction

Exercise

Prove that number restriction is an effective extension of the expressivity of $\mathcal{ALC}$, i.e., show that that $\mathcal{ALC}$ is strictly less expressive than $\mathcal{ALCN}$.

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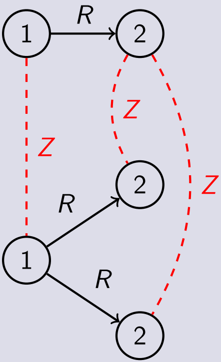

$\models (\leq 1)R$

$\not\models (\leq 1)R$

Qualified number restriction

Exercise

Prove that qualified number restriction is an effective extension of the expressivity of $\mathcal{ALCN}$, i.e., show that that $\mathcal{ALCN}$ is strictly less expressive than $\mathcal{ALCQ}$.

Solution (outline)

1. *Extend the notion of bisimulation relation to $\mathcal{ALCN}$.*
2. *Prove that $\mathcal{ALCN}$ is bisimulation invariant for the bisimulation relation defined in 1*
3. *Prove that $\mathcal{ALCQ}$ is more expressive than $\mathcal{ALCN}$.*

Bisimulation for $\mathcal{ALCN}$

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A $\mathcal{ALCN}$-bisimulation ρ between two $\mathcal{ALCN}$ interpretations $\mathcal{I}$ and $\mathcal{J}$ is a bisimulation ρ, that satisfies the following additional condition when $d \rho e$:

relation (cardinality) equivalence

- if $d_1, \ldots, d_n$ are all the distinct elemnts of $\Delta^{\mathcal{I}}$ such that $\langle d, d_i \rangle \in R^{\mathcal{I}}$ for $1 \leq i \leq n$, then there are exactly n, $e_1, \ldots, e_n$ elements of $\Delta^{\mathcal{J}}$ such that $(e, e_i) \in R^{\mathcal{J}}$ for all $1 \leq i \leq n$
- Same property in the opposite direction

$(\mathcal{I}, d) \sim (\mathcal{J}, e)$ means that there is a bisimulation ρ between $\mathcal{I}$ and $\mathcal{J}$ such that $e \rho e$.

Invariance w.r.t. $\mathcal{ALCN}$

Theorem

If $(\mathcal{I}, d) \sim (\mathcal{J}, e)$ *then for every* $\mathcal{ALCN}$ *concept* C $(\mathcal{I}, d) \models C$ *if and only if* $(\mathcal{J}, e) \models C$

Proof.

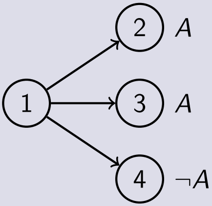
By induction on the complexity of C, similar as for $\mathcal{ALC}$ bisimulation with the following additional base step:

If C is $(\leq n)R$ If $(\mathcal{I}, d) \models (\leq n)R$, then there are $m \leq n$ elements $d_1, \ldots, d_m$ with $R(d, d_i)$. The additional condition on $\mathcal{ALCI}$-bisimulation implies that, there are exactly m elements $e_1, \ldots, e_m$, of $\Delta^{\mathcal{J}}$ such that $(e, e_i) \in R^{\mathcal{J}}$. which implies that $(\mathcal{J}, e) \models (\leq n)R$.

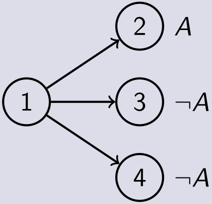
□

$\mathcal{ALCQ}$ is more expressive than $\mathcal{ALCN}$

We show that in $\mathcal{ALCQ}$ we can distinguish]two models which are not distinguishable in $\mathcal{ALCN}$



$\models (\leq 1)R.\neg A$



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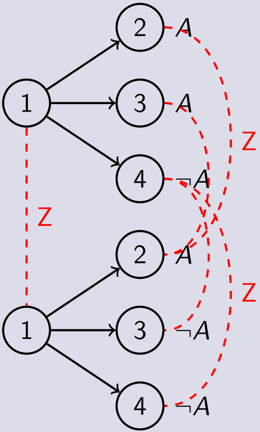

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Representing number restriction with inverse and functional roles

Exercise

Suppose that the concept C and T-box $\mathcal{T}$ contains number restrictions only on a single role R. Define set of axioms $\mathcal{T}_R$ such and a transformation τ from concepts of $\mathcal{ALCN}$ and $\mathcal{ALCIF}$ such that the following fact holds: C is satisfiable w.r.t. $\mathcal{T}$ in $\mathcal{ALCN}$ iff $\tau(C)$ is satisfiable w.r.t. $\tau(\mathcal{T}) \cup \mathcal{T}_R$ in $\mathcal{ALCIF}$

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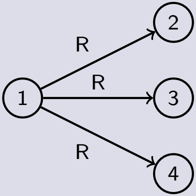
Intuitive solution

Replace the role R with $R_1, \ldots, R_n$ used for counting the number of R's successors.



$1 \models (\leq 3)R$
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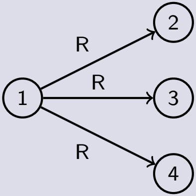

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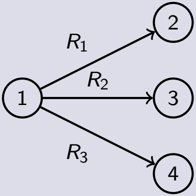

$1 \models \exists R_1.\top$
$1 \models \exists R_2.\top$
$1 \models \exists R_3.\top$
$1 \models \neg\exists R_4.\top$

Encoding number restriction with inverse and functional roles

Solution (Formal)

1. *n is the maximum number occurring in a number restriction of C*
2. *for every role R introduce* $R_1, \dots, R_{n+1}$
3. *for every role* R_i, $\mathcal{T}_R$ *contains the axioms:*
   1. $\exists R_{i+1}.\top \sqsubseteq \exists R_i.\top$ *for* $1 \leq i \leq n$
   2. $\top \sqsubseteq (\leq 1)R_i$ *for* $1 \leq i \leq n$ *(NB:* R_{n+1} *is not functional)*
   3. $\top \sqsubseteq \forall R_i.(\forall R_j^-.\bot)$ *for* $1 \leq i \neq j \leq n$
4. $\tau((\geq m)R) = \exists R_m.\tau(A)$
5. $\tau((\leq m)R) = \forall R_{m+1}.\neg\tau(A)$
6. $\tau(\exists R.A) = \exists R_1.\tau(A) \sqcup \dots \sqcup \exists R_{n+1}.\tau(A)$
7. $\tau(\forall R.A) = \forall R_1.\tau(A) \sqcap \dots \sqcap \forall R_{n+1}.\tau(A)$

Encoding number restriction with inverse and functional roles

Solution (Formal (cont'd))

We have to prove that if C is satisfiable, then $\tau(C)$ is satisfiable in $\mathcal{T}_R$.

1. *If C is satisfiable in $\mathcal{ALCN}$, then it has a tree-shaped model $\mathcal{I}$*

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2. *Extend $\mathcal{I}$ into $\mathcal{J}$ with the interpretation of $R_1, \ldots, R_{n+1}$ as follows. For all $d \in \Delta^{\mathcal{I}}$, let $R^{\mathcal{I}}(d) = \{d_1, \ldots, d_m, \ldots\}$ is the set of R-successors of d in $\mathcal{I}$, then*

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   - *if $|D| < n$, then add (d, d_i) to $R_i^{\mathcal{J}}$ for $1 \leq i \leq |D|$*

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   - *if $|D| \geq n$, then add (d, d_i) to $R_i^{\mathcal{I}}$ for $1 \leq i \leq n$ and also add (d, d_j) to $R_{n+1}^{\mathcal{I}}$ for $j \geq n+1$*

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We have to prove that if C is satisfiable, then $\tau(C)$ is satisfiable in $\mathcal{T}_R$.

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2. *Extend $\mathcal{I}$ into $\mathcal{J}$ with the interpretation of $R_1, \ldots, R_{n+1}$ as follows. For all $d \in \Delta^{\mathcal{I}}$, let $R^{\mathcal{I}}(d) = \{d_1, \ldots, d_m, \ldots\}$ is the set of R-successors of d in $\mathcal{I}$, then*
    - *if $|D| < n$, then add (d, d_i) to $R_i^{\mathcal{J}}$ for $1 \leq i \leq |D|$*
    - *if $|D| \geq n$, then add (d, d_i) to $R_i^{\mathcal{I}}$ for $1 \leq i \leq n$ and also add (d, d_j) to $R_{n+1}^{\mathcal{I}}$ for $j \geq n+1$*
3. *Prove that $\mathcal{J}$ is a model of $\mathcal{T}_R$*

Encoding number restriction with inverse and functional roles

Solution (Formal (cont'd))

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3. *Prove that $\mathcal{J}$ is a model of $\mathcal{T}_R$*
4. *Prove that $\mathcal{J}$ is a model of $\tau(C)$*

Encoding number restriction with inverse and functional roles

Solution (Formal (cont'd))

Finally we have to prove that if $\tau(C)$ is satisfiable in $\mathcal{T}_R$, then C is satisfiable.

1. *Let $\mathcal{J}$ be a tree-shaped model of $\mathcal{T}_R$ that satisfies C.*
2. *Let $\mathcal{I}$ be obtained by extending $\mathcal{J}$ with the interpretation of R as follows $R^{\mathcal{I}} = R_1^{\mathcal{I}} \cup \cdots \cup R_{n+1}^{\mathcal{I}}$*
3. *prove by induction on C, that $\mathcal{I}$ is a model of C.*

Role hierarchy $\mathcal{H}$

Definition

Role Hierarchy A role hierarchy $\mathcal{H}$ is a finite set of role subsumptions, i.e., expressions of the form

$$R \sqsubseteq S$$

for role symbols R and S We say that R is a subrole of S

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Exercise

Explain why the construct $R \sqsubseteq S$ cannot be axiomatized by the subsumptions

$$\exists R.\top \sqsubseteq \exists S.\top$$
$$\forall S.\top \sqsubseteq \forall R.\top$$

Transitive roles $\mathcal{S}$

Semantic condition

$\mathcal{I} \models tr(R)$ if $R^{\mathcal{I}}$ is a transitive relation.

Exercise

Explain why transitive roles cannot be axiomatized by the axiom

$$\exists R.(\exists R.A) \sqsubseteq \exists R.A$$

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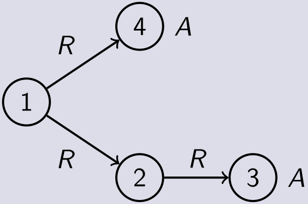
Solution



this model satisfies the axiom $\exists R.(\exists R.A) \sqsubseteq \exists R.A$ but R is not transitive

T-box internalization

Satisfiability w.r.t. T-box vs. concept satisfiability

Until now we have distinguished between the following two problems:

- Satisfiability of a concept C and
- Satisfiability of a concept C w.r.t. a T-box $\mathcal{T}$.

Clearly the first problem is a special case of the second, but with expressive languages that support role hierarchy and transitive role satisfiability w.r.t., T-box can be reduced to satisfiability.

This is like in propositional or first order logic where the problem of checking $\Gamma \models \phi$ (validity under a finite set of axioms Γ) reduces to the problem of checking the validity of a single formula. I.e., $\bigwedge \Gamma \rightarrow \phi$.

T-box internalization for logics stronger than $\mathcal{SH}$

Lemma

Representing the whole t-box in a single concept Let C a concept and $\mathcal{T} = \{A_1 \sqsubseteq B_1, \ldots, A_n \sqsubseteq B_n\}$ be a finite set of GCI.

$$C_{\mathcal{T}} = \sqcap_{i=1}^{n} \neg A_i \sqcup B_i$$

Let U be a new transitive role, and let

$$\mathcal{R}_U = \{R \sqsubseteq U | \textit{for all role } R \textit{ appearing in } C \textit{ and } \mathcal{T}\}$$

C is satisfiable w.r.t., $\mathcal{T}$ iff $C \sqcap C_{\mathcal{T}} \sqcap \forall U.C_{\mathcal{T}}$ is satisfiable w.r.t. $\mathcal{R}_U$