Logics for Data and Knowledge Representation

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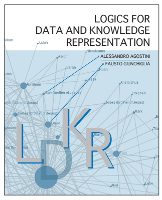
The Logic of Predicates



- Introduction
- Syntax
- Semantics
  - models
  - entailment
- Reasoning
  - reduction to PL

Properties: Normal Form Theorems

Normal Form Theorems

- A normal form theorem basically tell us that for every formula F of a given logic, there is a formula F* of some special syntactic form and *in the same logic* such that F and F* are logically equivalent, i.e. have the same models
- For F and F* be PL propositions or first-order formulas then we would write:
  $\models F \leftrightarrow F^*$.

Normal Form Theorems in FOL

- We present three normal form theorems, each considering logical equivalence of a FO-formula F with a formula F* in one of the following "special syntactic," *normal* forms:
  - (full) conjunctive/disjunctive (CNF/DNF)
  - prenex (PNF)
  - prenex CNF/DNF (PCNF/PDNF)

Quantifier-Free (QF) Sentences

- Definition. A FO-formula is quantifier-free if it contains no occurrence of $\forall x$ or $\exists x$.
- Definition. A quantifier-free sentence is a (first-order) sentence that is quantifier-free.
- Definition. Let $\alpha_1,\ldots,\alpha_n$ be FO-formulas. A truth-functional compound of $\alpha_1,\ldots,\alpha_n$ is a FO-formula built up from $\alpha_1,\ldots,\alpha_n$ using only $\neg$, $\wedge$, and $\vee$, without quantifiers.

CNF and DNF for QF-Sentences

- Definition. A truth-functional compound of wffs $\alpha_1,\ldots,\alpha_n$ is in conjunctive normal form (CNF) if it is a conjunction of disjunctions of wffs from the $\alpha_1,\ldots,\alpha_n$ and their negations.
- Definition. A truth-functional compound of wffs $\alpha_1,\ldots,\alpha_n$ is in disjunctive normal form (DNF) if it is a disjunction of conjunctions of wffs from the $\alpha_1,\ldots,\alpha_n$ and their negations.

Normal Form Theorem for CNFs in FOL

- Theorem (CNF Theorem). Every truth-functional compound of given formulas is logically equivalent to one that is in CNF.
- *Proof*:
  It is based on the same (effective) procedure used in propositional logic to convert a proposition into its CNF. Such procedure, when applied to truth-functional compounds (which don't contain implication symbols) is based on the following rules (logical equivalences):

Conversion to CNF Rules

- $\neg\neg\alpha$ is replaced by α
  rule based on $\models \neg\neg\alpha \leftrightarrow \alpha$ double-negation elimination
- $\neg(\alpha \wedge \beta)$ is replaced by $(\neg\alpha\vee\neg\beta)$
  $\neg(\alpha \vee \beta)$ is replaced by $(\neg\alpha\wedge\neg\beta)$

  rules based on $\models \neg(\alpha \wedge \beta) \leftrightarrow (\neg\alpha\vee\neg\beta)$
  $\models \neg(\alpha \vee \beta) \leftrightarrow (\neg\alpha\wedge\neg\beta)$
  De Morgan's Law for $\wedge$
  De Morgan's Law for $\vee$
- continue...

Conversion to CNF Rules (cont')

- $(\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ is replaced by $(\alpha \wedge (\beta \vee \gamma))$

  rule based on $\models (\alpha \wedge (\beta \vee \gamma)) \leftrightarrow ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$

  distributivity of $\wedge$ over $\vee$
- $\alpha \vee (\beta \wedge \gamma)$ is replaced by $(\alpha \vee \beta) \wedge (\alpha \vee \gamma)$

  rule based on $\models (\alpha \vee (\beta \wedge \gamma)) \leftrightarrow ((\alpha \vee \beta) \wedge (\alpha \vee \gamma))$

  distributivity of $\vee$ over $\wedge$

Example

- Let's convert to its CNF the formula: $\neg(\alpha \vee (\neg\beta \wedge \gamma))$.
- By using the rules above we have:
  $\models \neg(\alpha \vee (\neg\beta \wedge \gamma))$ iff
  $\models \neg\alpha \wedge \neg(\neg\beta \wedge \gamma)$ iff
  $\models \neg\alpha \wedge (\neg\neg\beta \vee \neg\gamma)$ iff
  $\models \neg\alpha \wedge (\beta \vee \neg\gamma)$.
- Observe that $\neg\alpha \wedge (\beta \vee \neg\gamma)$ is in CNF.

Normal Form Theorem for DNFs in FOL

- Theorem (DNF Theorem). Every truth-functional compound of given formulas is logically equivalent to one that is in DNF.
- *Proof*:
  Similar to the proof of CNF Theorem, with the exception that distributive laws for $\wedge$ and $\vee$ are used differently, i.e.:
  - $(\alpha \wedge (\beta \vee \gamma))$ is replaced by $(\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$
  - $(\alpha \vee \beta) \wedge (\alpha \vee \gamma)$ is replaced by $\alpha \vee (\beta \wedge \gamma)$.

  QED

Prenex Normal Form (PNF)

- Definition. A wff is a prenex formula, or in prenex normal form (or a quantified Boolean formula, QBF), if it is of this form:

  $Q_1x_1\ Q_2x_2 \ldots Q_kx_k\ \alpha(x_1, x_2, \ldots, x_k)$,
  where each Q_ix_i is a universal or existential quantifier, x_i is different from x_j for $i \neq j$ and $\alpha(x_1, x_2, \ldots, x_k)$ contains no quantifiers (quantifier-free).

Prenex Normal Form Remarks and Example

- Remark 1.
  $Q_1x_1\ Q_2x_2 \ldots Q_kx_k$ is called the prefix;
  $\alpha(x_1, x_2, \ldots, x_k)$ is called the matrix.
- Remark 2. For k=0, a prenex formula
  $Q_1x_1\ Q_2x_2 \ldots Q_kx_k\ \alpha(x_1, x_2, \ldots, x_k)$
  contains no quantifiers at all.
- Example.
  $\forall x_1 \exists x_3 \forall x_7 \exists x_2\ P(f(x_1), g(x_1, x_3), x_5)$

Normal Form Theorem for PNFs

- Theorem (PNF Theorem). Every FO-formula is logically equivalent to one that is in PNF.
- *Proof*:
  We need to seek a prenex formula logically equivalent to an arbitrary FO-formula. In seeking such formula in PNF, we need to pull quantifiers in the original formula out (leftmost). We list the rules (logical equivalences) which allow us to do this in general. The following rules are used together with other rules (as e.g., ¬¬-elimination) we have already presented (cf. CNF/DNF Theorems). ... continue...

Conversion to PNF Rules

- $\neg\forall x\alpha$ is replaced by $\exists x\neg\alpha$
  rule based on $\models \neg\forall x\alpha \leftrightarrow \exists x\neg\alpha$

  syntactic definition of ∃ by ∀ and ¬
- $\neg\exists x\alpha$ is replaced by $\forall x\neg\alpha$
  rule based on $\models \neg\exists x\alpha \leftrightarrow \forall x\neg\alpha$
  i.e. $\models \exists x\alpha \leftrightarrow \neg\forall x\neg\alpha$

  syntactic definition of ∃ by ∀ and ¬

Conversion to PNF Rules (cont')

- $\forall x\alpha(x) \wedge \forall y\beta(y)$ is replaced by $\forall u\ (\alpha(u) \wedge \beta(u))$
  rule based on $\models (\forall x\alpha(x) \wedge \forall y\beta(y)) \leftrightarrow \forall u\ (\alpha(u) \wedge \beta(u))$
- $\forall x\alpha(x) \vee \forall y\beta(y)$ is replaced by $\forall u \forall v\ (\alpha(u) \vee \beta(v))$
  rule based on $\models (\forall x\alpha(x) \vee \forall y\beta(y)) \leftrightarrow \forall u \forall v\ (\alpha(u) \vee \beta(v))$
- $\exists x\alpha(x) \wedge \exists y\beta(y)$ is replaced by $\exists u \exists v\ (\alpha(u) \wedge \beta(v))$
  rule based on $\models (\exists x\alpha(x) \wedge \exists y\beta(y)) \leftrightarrow \exists u \exists v\ (\alpha(u) \wedge \beta(v))$
- $\exists x\alpha(x) \vee \exists y\beta(y)$ is replaced by $\exists u\ (\alpha(u) \vee \beta(u))$
  rule based on $\models (\exists x\alpha(x) \vee \exists y\beta(y)) \leftrightarrow \exists u\ (\alpha(u) \vee \beta(u))$

  distributivity of quantifiers over ∧ and ∨

Prenex *Conjunctive* Normal Form (PCNF)

- Definition. A wff is a prenex conjunctive formula, or in prenex conjunctive normal form (PCNF) if it is a prenex formula whose matrix is in conjunctive normal form.
- Example.
  $\forall x_1 \exists x_3 \forall x_7 \exists x_2\ (\neg\alpha(x_1) \wedge (\beta(x_1, x_3) \vee \neg\gamma(x_5)))$

Normal Form Theorem for PCNFs

- Theorem (PCNF Theorem). Every FO-wff is logically equivalent to one that is in PCNF.
- *Proof*:
  First, convert the given FO-formula in its equivalent PNF. Second, work on the matrix of such formula in PNF and convert it to its equivalent CNF. The rules of conversion are given as in the proofs of CNF/DNF/PNF Theorems. QED
- A PDNF theorem holds for disjunctive PFs.

Correspondence Theorem PL-FOL *"in the finite"*

Correspondence Theorems

- A correspondence theorem is a normal form theorem across two (or more) logics.
- It basically tell us that for every formula F of a given logic, there is a formula F* of some special syntactic form written *in an expanded language* such that F and F* are logically equivalent, i.e. have the same models.
- Below we consider expansion by constants.

Correspondence PL-FOL "in the Finite"

- We present a correspondence theorem between propositional logic and first-order logic in the case a first-order language is interpreted over *finite* domains.
- The theorem will be given as a corollary of a normal form theorem saying that every formula of a FO-language *finitely interpreted* is logically equivalent to a quantifier-free wff of some normal form in an expanded language.

Finite Quantification Normal Form (FQNF)

- Let L^+ be an expansion of L obtained by adding to L's alphabet of nonlogical symbols the set $\{c_1,...,c_n\}$ of individual constants.
- Let $\alpha=\forall xP(x)$, $\beta=\exists xP(x)$ be sentences of L.

  (a) The finite quantifier-free normal form of α in L^+ is the L^+-sentence $P(c_1) \wedge ... \wedge P(c_n)$.

  (b) The finite quantifier-free normal form of β in L^+ is the L^+-sentence $P(c_1) \vee ... \vee P(c_n)$.

Conversion to FQNF Rules

- $\forall x\ \alpha(x)$ (in L) is replaced by $\alpha(c_1) \wedge ... \wedge \alpha(c_n)$ (in L^+) rule based on $\models \forall x\ \alpha(x) \leftrightarrow (\alpha(c_1) \wedge ... \wedge \alpha(c_n))$

  $\forall$-elimination over finite domains

- $\exists x\ \alpha(x)$ (in L) is replaced by $\alpha(c_1) \vee ... \vee \alpha(c_n)$ (in L^+) rule based on $\models \exists x\ \alpha(x) \leftrightarrow (\alpha(c_1) \vee ... \vee \alpha(c_n))$

  $\exists$-elimination over finite domains

- Remark: These replacements apply recursively. (See the next examples)

Normal Form (FQNF) Example 1

- Let L^+ be an expansion of L obtained by adding the set of individual constants $\{c_1,c_2\}$.
- Let $\alpha = \forall x \exists y P(x,y)$ be a L-sentence.

  - First, eliminate $\exists y$ and y:

  $\forall x\ (P(x,c_1) \lor P(x,c_2))$

  - Then, eliminate $\forall x$ and x:

  $(P(c_1,c_1) \lor P(c_1,c_2)) \land (P(c_2,c_1) \lor P(c_2,c_2))$

Normal Form (FQNF) Example 1a

- Let L^+ be an expansion of L obtained by adding the set of individual constants $\{c_1,c_2\}$.
- Let $\alpha = \forall x \exists y P(x,y)$ be a L-sentence.

  - Now, *first* eliminate $\forall x$ and x:

  $\exists y P(c_1,y) \land \exists y P(c_2,y)$

  - Then, eliminate $\exists y$ and y:

  $(P(c_1,c_1) \lor P(c_1,c_2)) \land (P(c_2,c_1) \lor P(c_2,c_2))$

Normal Form (FQNF) Example 2

- Let L^+ be an expansion of L obtained by adding the set of individual constants $\{c_1,c_2\}$.
- Let $\alpha = \forall x \forall y P(x,y)$ be a L-sentence.

  - First, eliminate $\forall y$ and y:

  $\forall x\ (P(x,c_1) \land P(x,c_2))$

  - Then, eliminate $\forall x$ and x:

  $(P(c_1,c_1) \land P(c_1,c_2)) \land (P(c_2,c_1) \land P(c_2,c_2))$

Normal Form (FQNF) Example 2a

- Let L^+ be an expansion of L obtained by adding the set of individual constants $\{c_1,c_2\}$.
- Let $\alpha = \forall x \forall y P(x,y)$ be a L-sentence.

  - Now, *first* eliminate $\forall x$ and x:

  $(\forall y P(c_1,y) \land \forall y P(c_2,y))$

  - Then, eliminate $\forall y$ and y:

  $(P(c_1,c_1) \land P(c_1,c_2)) \land (P(c_2,c_1) \land P(c_2,c_2))$

Normal Form (FQNF) FQNF for α is Unique

- The foregoing examples illustrate that there is a *unique* finite quantifier-free normal form α_{FQNF} of a given L-formula α in a fixed expansion L^+. More generally we have:
- Theorem. Let L^+ be an expansion of L. Every L-sentence α has a unique α_{FQNF} in L^+ (modulo the commutativity of $\land$ and $\lor$).

Finite Universe for a FO-Language

- Definition. Let L be a FO-language.

  (a) $M(L)^{fin}$ is the set of all L-structures whose domain is finite.

  (b) The finite universe for L (i.e., U_L) is the *finite* union of the Ds s.t. (D,I) is in $M(L)^{fin}$.
- In the following, let us assume L, $M(L)^{fin}$, $U_L = \{a_1,\ldots,a_n\}$ and L's expansion L^+ be fixed.

Finite Satisfiability

- Let L-structure M = (D,I) be given.

1. M finitely satisfies (in U_L) a L-sentence α if M satisfies α and M = (U_L,I) for any I.
2. α is finitely valid (in U_L) (written $\models^f \alpha$, or also $\models^U \alpha$) if M $\models \alpha$ for every M = (U_L,I).
3. We say α and β are logically equivalent in the finite (in U_L) if $\models^f \alpha\leftrightarrow\beta$ ($\models^U \alpha\leftrightarrow\beta$).

Finite Quantification Normal Form Theorem

- Theorem (FQNF Theorem). Every sentence of L is logically equivalent in the finite to one that is in finite quantifier-free normal form.

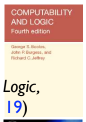
Finite Correspondence Theorem PL-FOL

- Corollary (Finite Correspondence PL-FOL). Every first order sentence is logically equivalent in the finite to a proposition.

  Proof: omitted.

- Remark. In fact the result above states a *reduction* of FOL to PL.

Some References



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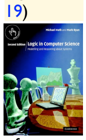

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