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Comparing formal theories of context in AI

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Abstract

The problem of context has a long tradition in different areas of artificial intelligence (AI). However, formalizing context has been widely discussed only since the late 80s, when J. McCarthy argued that formalizing context was a crucial step toward the solution of the problem of generality. Since then, two main formalizations have been proposed in AI: *Propositional Logic of Context* (PLC) and *Local Models Semantics/MultiContext Systems* (LMS/MCS). In this paper, we propose the first in depth comparison between these two formalizations, both from a technical and a conceptual point of view. The main technical result of this paper is the formal proof of the following facts: (i) that PLC can be embedded into a particular class of MCS, called MPLC; (ii) that MCS/LMS cannot be embedded in PLC using only lifting axioms to encode bridge rules, and (iii) that, under some important restrictions (including the hypothesis that each context has finite and homogeneous propositional languages), MCS/LMS can be embedded in PLC with generic axioms. The last part of the paper contains a comparison of the epistemological adequacy of PLC and MCS/LMS for the representation of the most important issues about contexts.

Keywords: Context; Contextual reasoning; Logic of context; Local models semantics; MultiContext systems; Propositional logic of context

1. Introduction

The study of a formal notion of context has a long history in different areas of artificial intelligence[1] (AI). Perhaps the first reference can be traced back to R. Weyhrauch and his

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[1] The interest in context is not limited to AI, though. On the contrary, it is discussed and used in various disciplines that are concerned with a theory of representation. In philosophy of language, the notion of pragmatic

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work on mechanizing logical theories in an interactive theorem prover called FOL [30]. However, it became a widely discussed issue only in the late 1980s, when J. McCarthy proposed the formalization of context as a crucial step toward the solution of the problem of generality: "When we take the logic approach to AI, lack of generality shows up in that the axioms we devise to express common sense knowledge are too restricted in their applicability for a general common sense database [...] Whenever we write an axiom, a critic can say that the axiom is true only in a certain context. With a little ingenuity the critic can usually devise a more general context in which the precise form of the axiom doesn't hold" [22]. This way, McCarthy raised the issue that no formal theory of common sense can get by without some formalization of context, as the representation of common sense axioms seems to crucially depend on the context in which they are asserted.

McCarthy elaborated his position in his notes on formalizing context [23]. In that paper, several important concepts (such as the notion of contexts as first class objects, the formula $ist(c, p)$—intuitively, that the formula p *is true* in the context c—the operations of *entering* and *exiting* contexts) were introduced. At around the same time, D. Lenat and R. Guha introduced an explicit mechanism of contexts in CYC, the biggest and perhaps most ambitious common sense knowledge base ever built. In his Ph.D. dissertation, Guha—under McCarthy's supervision—proposed a first formalization of context along the lines suggested in [23]. In particular, Guha introduced a formal semantics for the formula $ist(c, p)$, and discussed several important concepts, such as the notion of context structure and vocabulary, the distinction between grammaticality (expressions that are well-formed in a sort of universal language) and meaningfulness (expressions that have a meaning in given context), and the notion of *lifting axioms* (namely, axioms relating the truth of formulas in different contexts); in addition, he discussed several applications and techniques of context-based problem-solving techniques (e.g., *lift-and-solve*). McCarthy and Guha's work was the starting point of Buvač and Mason's *Propositional Logic of Context* (PLC) [10]. PLC explicitly aimed at formalizing McCarthy's intuitions on context, while giving a more traditional, modal flavor to Guha's semantics. A particular relevance is given to the idea that contexts must be formalized as first class objects (i.e., the logical language must contain terms for contexts, and the interpretation domain contains objects for contexts), and to the mechanisms of entering and exiting a context, which are identified as the two main mechanisms of contextual reasoning. [9] is a generalization of PLC to first-order languages.

Following a different line of thought, in the early 90s F. Giunchiglia proposed a different approach to the problem of context and an original formalization. In his 1993 paper on

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context has been used to provide a semantics to indexical (demonstrative) languages at least since Y. Bar-Hillel's seminal paper on indexical expressions [3]. Almost twenty years later, D. Kaplan published in the *Journal of Philosophical Logic* his well-known formalisation of a logic of demonstratives [21]. A broader philosophical approach to context was proposed and developed by J. Perry in his papers on indexicals and demonstratives, see [26]. Another approach, based on situation semantics, was pursued by J. Barwise and others [4,28]. Recently, R. Thomason has started working on a type-theoretic foundation of context [29]. In cognitive science, many authors have proposed theories of mental representation where mental contents are thought of as partitioned into multiple contexts (also called spaces [13], mental spaces [14], etc.). We only need to mention here that the notion of context is very important for other disciplines such as pragmatics, linguistics, formal ontology (see [1,5,7] for two recent collections of interdisciplinary papers on context).

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Contextual Reasoning [16], the formalization of context was motivated by the so-called *problem of locality*, namely the problem of modeling reasoning which uses only a subset of what reasoners know about the world. The idea is that, while solving a problem on a given occasion, people do not use all their knowledge, but construct a "local theory" (which corresponds to Giunchiglia's intuitive notion of context) and use it *as if* it contained all relevant facts about the problem at hand; while reasoning, people can switch from one context to another, for example when the original context is not adequate to solve the problem. In this approach, the emphasis is more on formalizing contextual reasoning than on formalizing contexts as first class objects. In [18], Giunchiglia and Serafini proposed MultiContext Systems (MCS) as a proof-theoretic framework for contextual reasoning; this paper introduces the notion of bridge rule, namely a special kind of inference rule whose premises and conclusion hold in different contexts. Recently, Ghidini and Giunchiglia proposed Local Models Semantics (LMS) as a model-theoretic framework for contextual reasoning, and used MCS to axiomatize many important classes of LMS [15]. From a conceptual point of view, Ghidini and Giunchiglia argued that contextual reasoning can be analyzed as the result of the interaction of two very general principles: the *principle of locality* (reasoning always happens in a context); and the *principle of compatibility* (there can be relationships between reasoning processes in different contexts). In other words, *contextual reasoning* is the result of the (constrained) interaction between distinct local structures.

PLC and LMS/MCS[2] are perhaps the most mature and well-known formalizations of context in AI.[3] Even though PLC and LMS/MCS are based on very different conceptual assumptions, and provide solutions which are technically very different, both aim at proposing a general solution to the problem of formalizing context, and at solving the fundamental issue of context in common sense reasoning. Quite surprisingly, however, so far the comparison between the two frameworks has been limited to a few lines of related work in the two groups respective papers. This paper aims at filling this gap, as it is the first in-depth investigation of the relationship between PLC and LMS/MCS, both from a conceptual and a technical point of view. The main technical result of this paper is a formal proof of the following facts: (i) that PLC can be embedded into a particular class of MCS, called MPLC; (ii) that LMS/MCS cannot be embedded in PLC using only lifting axioms to encode bridge rules, and (iii) that, under some important restrictions (including the hypothesis that each context has finite and homogeneous propositional languages), LMS/MCS can be embedded in PLC, but only if we allow also axioms which are not lifting axioms. Conceptually, we argue that the restrictions needed to prove the theorem have a significant impact on the fulfillment of the intuitive desiderata that were brought forward to motivate the formalization of context in AI. In particular, we argue that these restrictions are necessary because PLC fails to model a strong notion of contextual (local) vocabulary, the general notion of context-dependent truth, and the more general notion of contextual reasoning modeled by LMS/MCS.

[2] We use the abbreviation LMS/MCS to refer to the general framework for contextual reasoning which includes a model-theoretic (LMS) and a proof-theoretic (MCS) part.

[3] We refer the reader to [2] for an excellent discussion of the work on formalizing context in AI.

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The paper goes as follows. In the first part, we introduce the two formalisms we want to compare; for each of them we describe the underlying intuitions and the way these intuitions are modeled in the formal system. Then we present the technical comparison, and in particular the theorems in which we show to what extent and how one system can be embedded in the other. In the last part, we discuss the impact of the technical comparison on the adequacy of the two systems to capture the general desiderata of a logic of context.

2. Theories of context

PLC and LMS/MCS are not simply two alternative formalizations of context. Most of all, they are the outcome of two different conceptual views on what context and contextual reasoning are, and this fact is reflected by the choices that were made at a technical level. In this section we quickly review the two approaches, and prepare the ground for the technical comparison between the two systems.[4]

2.1. Propositional logic of context

The intuitions motivating PLC, described in various papers by McCarthy and his group (see, e.g., [9,10,20,22,23]), can be summarized as follows:

- contexts are first class objects, namely objects that belong to the domain of interpretation of a formal language. This means that the formal language of a theory of context should contain terms denoting contexts, and that it should allow one to predicate properties about these objects and to express relations between contexts (e.g., that a context is more general than another), or between contexts and other objects (e.g., that the time of a context c is t);
- a formula is always stated *in* a context. However, the same context can be described from different perspectives, i.e., the content of a particular context is itself context-dependent. So, for instance, in the context of the 1950s, the context of car racing is different than the context of car racing viewed from the today's context. This property, called *non-flatness*, is formalized by having each formula prefixed by a sequence $\kappa_1 \dots \kappa_n$ of context labels[5] (notationally, $\kappa_1 \dots \kappa_n : \phi$);
- a context is modeled as *a set* of truth assignments, each of which represents a possible state of the world as described in the context. This resembles the intuition behind possible world semantics. A formula ϕ is true (holds) in a context if the formula is assigned to true by every assignment associated to the context;
- a context is always partial, namely only a subset of what can be said is given an interpretation in each context. For instance in the context of the 1950s the sentence "John has a mobile phone" is not interpreted. So, even if PLC uses a traditional (modal)

[4] An exhaustive presentation of the two formalisms is beyond the scope of this paper; interested readers can refer to the bibliography for more details.

[5] Henceforth we will not stress the difference between context labels and contexts unless necessary to make clear what we are talking about.

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definition of formal language, the notion of contextual vocabulary is introduced by allowing *partial* truth assignments for each context. This corresponds to the intuition that there is a limited collection of facts that one can talk about in each context;

- statements about a context are stated in other contexts via so-called *ist*-formulas, i.e., formulas of the form $ist(\kappa, \phi)$. The formula $ist(\kappa, \phi)$ is read as "ϕ *is true* in the context κ". This formula, if asserted in a context κ', means that, viewed from κ', ϕ is true in κ.
- there is an intuitive relation between the assertions $\kappa'\kappa : \phi$ and $\kappa' : ist(\kappa, \phi)$. Indeed, the latter is true if the former is true, and vice versa. This is the main semantic property formalized in PLC. This property is axiomatized via an inference rule called CS (a contextual version of the modal rule of necessitation) that allows deriving $\kappa' : ist(\kappa, \phi)$ from $\kappa'\kappa : \phi$. This is the main contextual reasoning pattern allowed in PLC. Intuitively, it corresponds to McCarthy's notion of *exiting* (or *transcending*) context κ.
- Other relations between contexts can be stated through the *lifting axioms*. Lifting axioms are defined as "... axioms which relate the truth in one context to the truth in another context. Lifting is the process of inferring what is true in one context based on what is true in another context by the means of lifting axioms" [20,24]. Most of the examples of lifting axioms one can find in the literature are Horn clauses of the following form:

$$ist(\kappa_1, \phi_1) \wedge \cdots \wedge ist(\kappa_n, \phi_n) \supset ist(\kappa_{n+1}, \phi_{n+1}) \qquad (1)$$

  In this paper we will use the term *lifting axioms* to denote lifting axioms in Horn form. Like any other formula, lifting axioms are always stated in a context, called an outer context.
- There is no outermost context. Indeed, for any context κ, there is an outer context κ' from which κ can be described.

In this paper, we use the version of PLC presented in [10]. Given a set $\mathbb{K}$ of labels, intuitively denoting contexts, the language of PLC is a multi modal language on a set of atomic propositions $\mathbb{P}$ with the modality $ist(\kappa, \phi)$ for each context (label) $\kappa \in \mathbb{K}$. More formally, the set of well-formed formulae $\mathbb{W}$ of PLC, based on $\mathbb{P}$, is defined as:

$$\mathbb{W} := \mathbb{P} \cup (\neg\mathbb{W}) \cup (\mathbb{W} \supset \mathbb{W}) \cup ist(\mathbb{K}, \mathbb{W})$$

The other propositional connectives are defined as usual. If κ is a context, then the formula $ist(\kappa, \phi)$ can be read as: ϕ is true in the context κ. PLC allows describing how a context is viewed from another context. For this reason, PLC introduces the notion of context sequence. Let $\mathbb{K}^*$ denote the set of finite context sequences and let $\bar{\kappa} = \kappa_1 \ldots \kappa_n$ denote any (possible empty) element of $\mathbb{K}^*$. The context sequence $\kappa_1\kappa_2$ represents how context κ_2 is viewed from context κ_1. Therefore, the intuitive meaning of the formula $ist(\kappa_2, \phi)$ in the context κ_1 is that ϕ holds in the context κ_2, from the point of view of κ_1. Similar interpretation can be given to formulae in context sequences longer than 2.

A model for PLC associates a set of partial truth assignments to each context sequence and satisfiability is defined with respect to a context sequence.

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Definition 2.1. A *model* $\mathfrak{M}$ of PLC is a partial function which maps each context sequence $\bar{\kappa} \in \mathbb{K}^*$ into a set of partial truth assignments for $\mathbb{P}$.

$$\mathfrak{M} \in \left(\mathbb{K}^* \to_p \mathbf{P}(\mathbb{P} \to_p \{\text{true}, \text{false}\})\right)$$

where $A \to_p B$ denotes the set of *partial* functions from A to B and $\mathbf{P}(A)$ denotes the powerset of A.

The intuition is that partial truth assignments can be used to model contexts with different languages, i.e., sets of meaningful formulae. Intuitively, in a PLC-model $\mathfrak{M}$, a formula ϕ is meaningful in a context sequence $\bar{\kappa}$ (and therefore it belongs to $\bar{\kappa}$'s language) if $\mathfrak{M}(\bar{\kappa})$ is defined and for every assignment in $\mathfrak{M}(\bar{\kappa})$, it is always possible to determine the truth of ϕ. In this way, a model $\mathfrak{M}$ defines a vocabulary, denoted by $\mathsf{Vocab}(\mathfrak{M})$. $\mathsf{Vocab}(\mathfrak{M})$ is the function that associates to each context sequence $\bar{\kappa}$ a subset of $\mathbb{P}$ for which all the assignments in $\mathfrak{M}(\bar{\kappa})$ are defined. That is, $\langle \bar{\kappa}, p \rangle \in \mathsf{Vocab}(\mathfrak{M})$ if and only if $\mathfrak{M}(\bar{\kappa})$ is defined and, for all $\nu \in \mathfrak{M}(\bar{\kappa})$, $\nu(p)$ is defined (where ν is a truth assignment to atomic propositions).

Satisfiability and validity of formulae are defined only for those models that provides enough vocabulary, i.e., the vocabulary which is necessary to evaluate a formula in a context sequence. Each formula ϕ in a context sequence $\bar{\kappa}$ implicitly defines its vocabulary, denoted by $\mathsf{Vocab}(\bar{\kappa}, \phi)$, which intuitively consists of the minimal vocabulary necessary to build the formula ϕ in the context sequence $\bar{\kappa}$. More formally, $\mathsf{Vocab}(\bar{\kappa}, \phi)$ is recursively defined as follows:

$$\mathsf{Vocab}(\bar{\kappa}, p) = \left\{\langle \bar{\kappa}, p \rangle\right\}$$
$$\mathsf{Vocab}(\bar{\kappa}, \neg\phi) = \mathsf{Vocab}(\bar{\kappa}, \phi)$$
$$\mathsf{Vocab}(\bar{\kappa}, \phi \supset \psi) = \mathsf{Vocab}(\bar{\kappa}, \phi) \cup \mathsf{Vocab}(\bar{\kappa}, \psi)$$
$$\mathsf{Vocab}\left(\bar{\kappa}, ist(\kappa, \phi)\right) = \mathsf{Vocab}(\bar{\kappa}\kappa, \phi)$$

Definition 2.2 (*Satisfiability and validity in PLC*). Let ϕ and $\mathfrak{M}$ be a formula and a model, such that $\mathsf{Vocab}(\bar{\kappa}, \phi) \subseteq \mathsf{Vocab}(\mathfrak{M})$; ϕ is satisfied in $\mathfrak{M}$ by an assignment $\nu \in \mathfrak{M}(\bar{\kappa})$ (notationally $\mathfrak{M}, \nu \models_{\bar{\kappa}} \phi$) according to the following clauses:

1. $\mathfrak{M}, \nu \models_{\bar{\kappa}} p$ iff $\nu(p) = \text{true}$;
2. $\mathfrak{M}, \nu \models_{\bar{\kappa}} \neg\phi$ iff not $\mathfrak{M}, \nu \models_{\bar{\kappa}} \phi$;
3. $\mathfrak{M}, \nu \models_{\bar{\kappa}} \phi \supset \psi$ iff not $\mathfrak{M}, \nu \models_{\bar{\kappa}} \phi$ or $\mathfrak{M}, \nu \models_{\bar{\kappa}} \psi$;
4. $\mathfrak{M}, \nu \models_{\bar{\kappa}} ist(\kappa, \phi)$ iff for all $\nu' \in \mathfrak{M}(\bar{\kappa}\kappa)$, $\mathfrak{M}, \nu' \models_{\bar{\kappa}\kappa} \phi$;
5. $\mathfrak{M} \models_{\bar{\kappa}} \phi$ iff for all $\nu \in \mathfrak{M}(\bar{\kappa})$, $\mathfrak{M}, \nu \models_{\bar{\kappa}} \phi$;
6. $\models_{\bar{\kappa}} \phi$ iff for all PLC-model $\mathfrak{M}$, such that $\mathsf{Vocab}(\bar{\kappa}, \phi) \subseteq \mathsf{Vocab}(\mathfrak{M})$, $\mathfrak{M} \models_{\bar{\kappa}} \phi$.

If the precondition $\mathsf{Vocab}(\bar{\kappa}, \phi) \subseteq \mathsf{Vocab}(\mathfrak{M})$, does not hold then, neither $\mathfrak{M}, \nu \models_{\bar{\kappa}} \phi$ nor $\mathfrak{M}, \nu \models_{\bar{\kappa}} \neg\phi$.

ϕ is *valid* in a context sequence $\bar{\kappa}$ if $\models_{\bar{\kappa}} \phi$; ϕ is *satisfiable* in a context sequence $\bar{\kappa}$ if there is a PLC-model $\mathfrak{M}$ such that $\mathfrak{M} \models_{\bar{\kappa}} \phi$. A set of formulae T is satisfiable at a context sequence $\bar{\kappa}$ if there is a model $\mathfrak{M}$ such that $\mathfrak{M} \models_{\bar{\kappa}} \phi$ for all $\phi \in T$.

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(PL)	$\vdash_{\bar{\kappa}} \phi$ if ϕ is an instance of a classical tautology
(K)	$\vdash_{\bar{\kappa}} ist(\kappa, \phi \supset \psi) \supset ist(\kappa, \phi) \supset ist(\kappa, \psi)$
(Δ)	$\vdash_{\bar{\kappa}} ist(\kappa_1, ist(\kappa_2, \phi) \vee \psi) \supset ist(\kappa_1, ist(\kappa_2, \phi)) \vee ist(\kappa_1, \psi)$
(MP)	$\dfrac{\vdash_{\bar{\kappa}} \phi \quad \vdash_{\bar{\kappa}} \phi \supset \psi}{\vdash_{\bar{\kappa}} \psi}$
(CS)	$\dfrac{\vdash_{\bar{\kappa}\kappa} \phi}{\vdash_{\bar{\kappa}} ist(\kappa, \phi)}$

Fig. 1. Axioms and inference rules for PLC.

The proposed sound and complete Hilbert-style axiomatization for validity in PLC is reported in Fig. 1.

2.2. Local Models Semantics/MultiContext Systems

The intuitions underlying LMS/MCS are summarized in the following points.

- A context is primarily a subset of an individual global state, or—slightly more formally—a partial and approximate theory of the world from some individual's perspective [16]. The most typical example is the collections of facts that an individual uses to reason about a given problem. In [17], this idea is expressed by saying that contexts are not thought of as part of the structure of the world (*metaphysical context*), but rather as a way of structuring an individual's representation of the world (*cognitive context*).
- Reasoning mainly happens locally to a single context. The set of facts that an individual takes in to consideration in order to draw a conclusion, via deductive reasoning, is a small subset of his/her whole knowledge. This set contains those facts which are relevant to the problem (s)he wants to solve, i.e., the one which are in the context (s)he is currently using. In other words, problems must be reasoned about in an appropriate *problem-solving context* (see for instance [8,19,23,25]).
- However, the interesting part of a theory of contextual reasoning is that there are possible relations between local reasoning processes. This is because different contexts are not simply unrelated representations, but different representations of the same world. For example, two contexts may describe the same piece of world from the same perspective but at different level of detail; or may describe the same piece of the world, only from different perspectives. In LMS relations between different perspectives are represented via a *compatibility relation* between local interpretation associated with each context. The proof theoretic counterpart of compatibility relations are bridge rules, i.e., inference rules with premises and consequences in different contexts.
- Finally, an important intuition is that not only each context is partial, but also that, in general, the relationship between different contexts can be described only to a partial extent. In other words, no matter how much we know about the relationship between two contexts, in general we cannot fully "translate" one context into the other, as each one may encode assumptions which are not fully explicit. Therefore, contexts form a multiplicity of representations which are not reducible to a single, uniform representation of the world.

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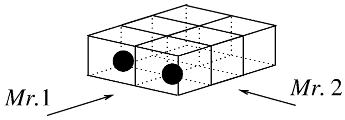

Fig. 2. The magic box.

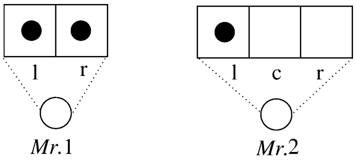

Fig. 3. Mr.1 and Mr.2's views.

In [15], these intuitions are synthesized into two general principles of contextual reasoning. The first, called the *principle of locality*, is meant to capture the idea that reasoning happens in partial, approximate, and perspectival representations of the world; the second, called the *principle of compatibility*, says that there exist constraints between reasoning processes in different contexts which guarantee their compatibility. In short, *contextual reasoning* is a collection of reasoning mechanisms that exploit the relationships among the local structures of different contexts.

A simple illustration of the intuitions discussed above can be given by introducing the so-called "magic box" example, proposed in [15], and depicted in Fig. 2. Two agents, Mr.1 and Mr.2 look at the magic box from different viewpoints. The box is "magic" because the observers cannot distinguish the depth inside it. Fig. 3 shows what Mr.1 and Mr.2 see in the scenario depicted in Fig. 2.

The views of Mr.1 and Mr.2 can be thought of as two different contexts. Both observers have a local representation of the box, which depends on their perspective. For example, Mr.1 sees a box with two slots while Mr.2 sees a box with three slots, or Mr.1 can see a ball in the left sector and one in the right, while Mr.2 sees only a single ball in the left sector. The two contexts are not independent of one another so that, for instance, if the context of Mr.1 contains the fact that there is a ball in the right slot, then the context of Mr.1 could not contain the statement that the box is empty. The relationships between contexts (local representations) lies at the heart of LMS/MCS, whose formalization we present next.

Let $\{L_i\}_{i\in I}$ be a family of languages defined over a set of indexes I (in the following we drop the index $i \in I$). Intuitively, each L_i is the (formal) language used to describe the facts in the context i. In this paper, we assume that I is (at most) countable. Let M_i be the class of all models (interpretations) of L_i. Each $m \in M_i$ is called a *local model* (of L_i).

A labeled formula of the kind $i:\phi$ is used to state that ϕ holds in i. As contexts have distinct languages, it may perfectly well be the case that $i:\phi$ is a formula, while $j:\phi$ is not (for some $j \neq i$). Conversely, it should be clear that the "same" formula in two distinct contexts is interpreted over different sets of local models, and therefore, in principle, have independent meaning. The meaning of $i:\phi$ is kept distinct from the meaning of $j:\phi$ (for $i \neq j$).

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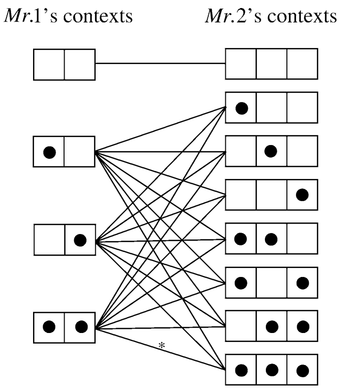

Fig. 4. Compatibility relation for the magic box. Each line connects two models that belongs to the same chain.

In the magic box example, the propositions l and r in the context 1 represent Mr.1's view that "there is a ball in the left sector" and "there is a ball in the right sector", respectively. Analogously, the propositions l, c, and r in the context 2 represent Mr.2's view that "there is a ball in the left (center and right) sector".

Definition 2.3 (*Compatibility chain*[6]). A *compatibility chain* $c = \{c_i \subseteq M_i\}_{i \in I}$ is a family of set of models of L_i such that each c_i is either empty or a singleton. c_i is the ith element of c. A compatibility chain is *nonempty* if at least one of its components is nonempty.

Definition 2.4 (*Compatibility relation and LMS-model*). A *compatibility relation* is a set of compatibility chains. A *LMS-model* is a compatibility relation that contains at least one nonempty compatibility chain.

A compatibility chain represents a set of "instantaneous snapshots of the world" each of which is taken from the point of view of the associated context. Due to the fact that contexts describe points of view of the *same world*, certain combinations of snapshots can never happen. In the magic box, for instance, the fact that Mr.1 and Mr.2 look at the same box, entails that if Mr.1 sees some ball, then Mr.2 sees some ball too. This relation is captured by the compatibility relation shown in Fig. 4.

Definition 2.5 (*Satisfiability and logical consequence*). Let C be a compatibility relation, $c \in C$ be a chain, ϕ a formula of L_i, and Γ a set of labelled formulas with labels different from i.

1. $c \models i : \phi$ if ϕ is true in all $m \in c_i$.[7]
2. $C \models i : \phi$ if for all $c \in C$, $c \models i : \phi$.

[6] For the sake of this paper, it is not necessary to introduce the more general definition of compatibility chain presented in [15].

[7] Notice that c_i can contain at most one model.

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3. $\Gamma \models_C i:\phi$, if for all $c \in C$, either $c \not\models j:\gamma$ for some $j:\gamma \in \Gamma$, or $c \models i:\phi$.
4. For any class of compatibility relations $\mathbf{C}$, $\Gamma \models_{\mathbf{C}} i:\phi$, if, for all models $C \in \mathbf{C}$, if for any $C \in \mathbf{C}$, $\Gamma \models_C i:\phi$.

We adopt the usual terminology of satisfiability and entailment for the statements about the relation $\models$.

MultiContext Systems (MCS) [18] are a class of proof systems for LMS.[8] The key notion in an MCS is that of bridge rule.

Definition 2.6 (*Bridge rule*). A *bridge rule br* on a set of indices I is a schema of the form:

$$\frac{i_1:A_1 \quad \ldots \quad i_n:A_n}{i:A}\,br$$

where $i_1,\ldots,i_n,i \in I$ and $A_1,\ldots,A_n,A$ are schematic formulae. A bridge rule can be associated with a *restriction*, namely a criterion which states the conditions of its applicability.

Examples of bridge rules are:

$$\frac{i:A}{j:A}\,i\text{-to-}j \qquad \frac{O:\mathit{Theorem}(\text{“}A\text{”})}{M:A}\,\mathcal{R}_{dn.} \qquad \frac{t:P(c)}{R:P(c,t)}\,\text{Reif}$$

The first bridge rule intuitively formalizes the fact that the context i is contained in (or is copied into) the context j. The bridge rule $\mathcal{R}_{dn.}$ formalize the fact that, in the (meta)context M, the predicate $\mathit{Theorem}(x)$ is a sound formalization of the provability in the (object) context O. Finally, the bridge rule “Reif” (for reification) reifies times in a reification context R, and it allows inferring that c is P at the time t (i.e., $P(c,t)$) from the fact that $P(c)$ holds in the context associated to the time t.

Definition 2.7 (*MultiContext System* (*MCS*)). A *MultiContext System* (*MCS*) for a family of languages $\{L_i\}$, is a pair $\text{MS} = \langle\{C_i = \langle L_i, \Omega_i, \Delta_i\rangle\}, \Delta_{br}\rangle$, where each $C_i = \langle L_i, \Omega_i, \Delta_i\rangle$ is a theory (on the language L_i, with axioms Ω_i and natural deduction inference rules Δ_i), and Δ_{br} is a set of bridge rules on I.

MCSs are a generalization of Natural Deduction (ND) systems [27]. The generalization amounts to using formulae tagged with the language they belong to. This allows for the effective use of the multiple languages. The deduction machinery of an MCS is the composition of two kinds of inference rules: *local rules*, namely the inference rules in each Δ_i, and *bridge rules*. Local rules formalize reasoning within a context (i.e., are only applied to formulae with the same index), while bridge rules formalize reasoning across different contexts.

Deductions in a MCS are trees of formulae which are built starting from a finite set of assumptions and axioms, possibly belonging to distinct languages, and by a finite number

[8] In this paper, we present a definition of MC system which is suitable for our purposes. For a fully general presentation, see [18].

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$$\dfrac{\dfrac{\dfrac{\dfrac{2:l}{2:l\vee c}\vee\mathrm{I}}{2:l\vee c\vee r}\vee\mathrm{I}}{1:l\vee r}\exists^{\bullet}_{2} \qquad \dfrac{\dfrac{1:\neg l \quad [1:l]}{1:\bot}\supset\mathrm{E}}{1:r}\bot \qquad [1:r]}{1:r}\vee\mathrm{E}$$

Fig. 5. Deduction for $2:l, 1:\neg l \vdash_{\mathrm{MBox}} 1:r$. The deduction starts from assuming the hypothesis $2:l$, in the context C_2. Then some local reasoning (two applications of $\vee$I rules) is performed, which allows us to draw an "exportable" conclusions. Namely a formula that is a premise of a bridge rule. This formula is then exported by means of the bridge rule $\exists^{\bullet}_{2}$, into the context C_1, where, via local reasoning we reach the conclusion $1:r$.

of applications of local rules and bridge rules. A formula $i:\phi$ is *derivable* from a set of formulae Γ in a MC system MS, in symbols, $\Gamma \vdash_{\mathrm{MS}} i:\phi$, if there is a deduction with bottom formula $i:\phi$ whose un-discharged assumptions are in Γ. A formula $i:\phi$ is a *theorem* in MS, in symbols $\vdash_{\mathrm{MS}} i:\phi$, if it is derivable from the empty set. The standard notation for deductions can be obtained by drawing a tree of labelled formulae. An example is shown in Fig. 5.

The MCS formalizing the magic box example (called MBox) is composed of two contexts 1 and 2 for Mr.1 and Mr.2 respectively. L_1 and L_2 are the propositional languages defined on the sets primitive propositions $\{l, r\}$ and $\{l, c, r\}$ respectively. The set of axioms of 1 and 2 are empty, as there is no restriction on the configuration of the box. Finally, the set of bridge rules between 1 and 2 are the following:

$$\frac{1:l\vee r}{2:l\vee c\vee r}\exists^{\bullet}_{1} \quad \frac{2:l\vee c\vee r}{1:l\vee r}\exists^{\bullet}_{2} \quad \frac{1:\neg(l\vee r)}{2:\neg(l\vee c\vee r)}\mathit{not}\,\exists^{\bullet}_{1} \quad \frac{2:\neg(l\vee c\vee r)}{1:\neg(l\vee r)}\mathit{not}\,\exists^{\bullet}_{2}$$

The bridge rule $\exists^{\bullet}_{1}$ formalizes the compatibility statement: "if Mr.1 sees a ball then Mr.2 sees a ball". The intuitive interpretation of the other bridge rules is similar. In Fig. 5, we propose an example of a deduction MBox proving that if Mr.2 sees a ball in the left sector ($2:l$) and Mr.1 does not sees any ball in the left sector ($1:\neg l$), then he sees one ball in the right sector ($1:r$).

3. Comparing PLC and LMS/MCS

PLC can be viewed as a multi-modal version of the Kripke's system K, extended with the axiom (Δ) (see Fig. 1). In [18], a family of MCS, called MBK, was proved to be equivalent to modal K; moreover, [15] presents the definition of a LMS for MBK (and the corresponding completeness result). To prove that PLC can be represented in LMS/MCS, we first show that vocabularies in PLC play no logical role. Then we extend MBK for multi-modal K, and we define the MultiContext System MPLC, in which (Δ) is a theorem.

Notice that the definition of satisfiability and validity in PLC given in [10] and reported in Definition 2.2, refers also to the vocabulary of a model. We show that an equivalent definition of satisfiability can be given in which such a parameter is dropped.

Let a *complete vocabulary* be a the vocabulary that associates to each context sequence the entire set of formulae $\mathbb{W}$.

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Theorem 3.1 (Reduction to complete vocabulary). *A formula is valid in* PLC *if and only if it is satisfied by all the* PLC*-models with complete vocabulary. Similarly, a formula is satisfiable in* PLC *if and only if there is a* PLC*-model with complete vocabulary that satisfies it.*

The proof of Theorem 3.1 follows by showing that each PLC-model that satisfies a formula ϕ can be extended to a PLC-model with a complete vocabulary satisfying ϕ. The complete proof (like most of the proofs of the other theorems of the paper) is in Appendix A.

PLC-models with complete vocabulary are equivalent to normal Kripke models in which: the set of worlds are the pair $\langle \nu, \bar{\kappa} \rangle$, the accessibility relation R_κ (for each $\kappa \in \mathbb{K}$) is defined as "$\langle \nu, \bar{\kappa}\kappa \rangle$ is accessible via R_κ from $\langle \nu, \bar{\kappa} \rangle$", and the truth assignment to $\langle \nu, \kappa \rangle$ is ν itself. Under this interpretation, Theorem 3.1 states that validity in PLC can be checked by considering a set of normal Kripke structure, and therefore that PLC is a normal modal logic.

3.1. Reconstructing PLC in LMS/MCS

To reconstruct PLC in MCS we start with the definition of the MCS corresponding to multi-modal K and then we add a suitable constraint for (Δ). For each (possibly empty) sequence $\bar{\kappa} \in \mathbb{K}^*$, the language $L_{\bar{\kappa}}$ is the smallest propositional language that contains $\mathbb{P}$ and the *atomic formula* $ist(\kappa, \phi)$ for any $\kappa \in \mathbb{K}$ and any formula $\phi \in L_{\bar{\kappa}\kappa}$. Notice that, unlike in PLC, the formula $ist(\kappa, \phi)$ is an atomic formula of $L_{\bar{\kappa}}$, and not the application of the modal operator $ist(\kappa, \ldots)$ to the formula ϕ.

Definition 3.1. An MBK($\mathbb{K}$)-model is a model for the family of languages $\{L_{\bar{\kappa}}\}_{\bar{\kappa} \in \mathbb{K}^*}$, such that, for any $c \in C$ and $\bar{\kappa}\kappa \in \mathbb{K}^*$:

1. if $c \models \bar{\kappa} : ist(\kappa, \phi)$, then $c \models \bar{\kappa}\kappa : \phi$;
2. if $c' \models \bar{\kappa}\kappa : \phi$ for all $c' \in C$ with $c_{\bar{\kappa}} = c'_{\bar{\kappa}}$, then $c \models \bar{\kappa} : ist(\kappa, \phi)$.

Definition 3.2. MBK($\mathbb{K}$) is a MCS on the family of languages $\{L_{\bar{\kappa}}\}_{\bar{\kappa} \in \mathbb{K}}$, where, for each $\bar{\kappa}$, $\Omega_{\bar{\kappa}}$ is empty and $\Delta_{\bar{\kappa}}$ is the set of propositional natural deduction inference rules, and Δ_{br} is the following set of bridge rules:

$$\frac{\bar{\kappa} : ist(\kappa, \phi)}{\bar{\kappa}\kappa : \phi}\mathcal{R}_{dn.\bar{\kappa}\kappa} \qquad \frac{\bar{\kappa}\kappa : \phi}{\bar{\kappa} : ist(\kappa, \phi)}\mathcal{R}_{up.\bar{\kappa}\kappa}$$

RESTRICTION $\mathcal{R}_{up.\bar{\kappa}\kappa}$ is applicable only if $\bar{\kappa}\kappa : \phi$ does not depend on any assumption with index $\bar{\kappa}\kappa$.

Soundness and completeness theorems for MBK($\mathbb{K}$) with respect to the class of MBK($\mathbb{K}$) models are given in [15]:

Theorem 3.2 (Soundness and completeness). $\Gamma \models_{\mathrm{MBK}(\mathbb{K})} \bar{\kappa} : \phi$ *if and only if* $\Gamma \vdash_{\mathrm{MBK}(\mathbb{K})} \bar{\kappa} : \phi$.

Theorem 3.2 is proved in [15] for the special case with $\mathbb{K}$ singleton (see Theorems B.1 and B.2 of [15]). The generalization for any $\mathbb{K}$, can be obtained by uniformly adding the indexes to such a proof.

Definition 3.3 (*MPLC-model*). An *MPLC-model* is an MBK($\mathbb{K}$)-model, that satisfies the following additional condition:

3. if $c \models \bar{\kappa}\kappa : ist(\kappa', \phi)$, then $c \models \bar{\kappa} : ist(\kappa, ist(\kappa', \phi))$.

Condition 3 is the compatibility constraint that corresponds to the axiom (Δ) in PLC.

Theorem 3.3. *Any* MBK($\mathbb{K}$)*-model C is an MPLC-model if and only if $C \models \bar{\kappa} : (\Delta)$, for every $\bar{\kappa}$.*

We modify MBK($\mathbb{K}$) in order to prove the axiom (Δ).

Definition 3.4 (*MPLC*). MPLC is an MCS defined as MBK($\mathbb{K}$) where the restriction of $\mathcal{R}_{up.\bar{\kappa}\kappa}$ is applied only if the premise of $\mathcal{R}_{up.\bar{\kappa}\kappa}$, is not of the form $\bar{\kappa}\kappa : ist(\kappa', \psi)$.

Now, we need to prove that the extension of MBK($\mathbb{K}$) is the right one, namely that MPLC is sound and complete w.r.t. the class of MPLC-models.

Theorem 3.4 (Soundness and completeness of MPLC). *MPLC is sound and complete w.r.t. the set C_{MPLC} of MPLC-models. In symbols*

$$\Gamma \vdash_{\mathrm{MPLC}} \bar{\kappa} : \phi \quad \textit{if and only if} \quad \Gamma \models_{C_{\mathrm{MPLC}}} \bar{\kappa} : \phi$$

Finally, we need to state the equivalence between MPLC and PLC w.r.t. provability.

Theorem 3.5 (MPLC is equivalent to PLC). $\vdash_{\bar{\kappa}} \phi$ *iff* $\vdash_{\mathrm{MPLC}} \bar{\kappa} : \phi$.

3.2. Reconstructing LMS/MCS in PLC

Before we proceed to compare the two logical systems, we observe that such a comparison is possible only of we introduce the following restrictions on MCS:

1. we must consider only MCSs with homogeneous languages in each context, as PLC does not properly support different vocabularies (see Theorem 3.1);
2. we restrict the comparison to MCSs in which all contexts have the same inference engine, which is contexts are all classical propositional theories;
3. for the sake of this comparison, we consider only ground bridge rules, i.e., bridge rules formulated using formulas of the languages and not schemas.

The general strategy to encode an MCS into PLC is shown in Fig. 6. Given a MCS with I contexts, we define a PLC with I contexts (one for each context in MCS) and an

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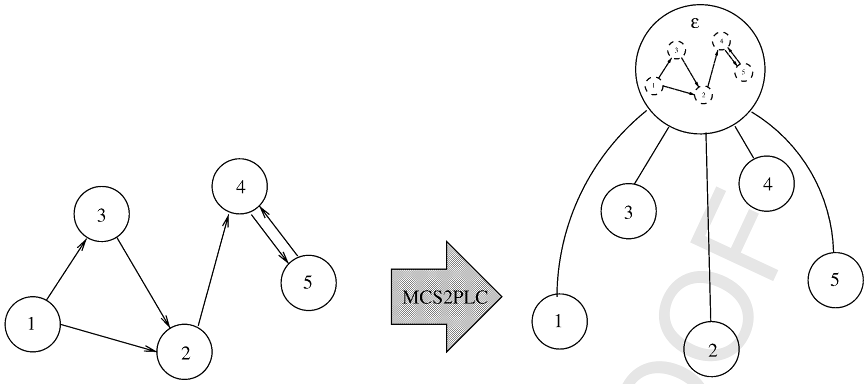

Fig. 6. From MCS to PLC.

additional meta-context ε. In ε, the content of each context and the compatibility relations (bridge rules) between contexts are described via *ist*-formulas.

The representation of the content of the MCS contexts is quite straightforward: any formula $i:\phi$ in MCS is translated into the formula $\varepsilon : ist(i,\phi)$ in PLC. For bridge rules, the translation is not so straightforward. Indeed, the first natural idea is to translate each bridge rule

$$\frac{i_1:\phi_1 \quad \dots \quad i_n:\phi_n}{i:\phi}$$

of an MCS into the lifting axiom $ist(i_1,\phi_1)\wedge\cdots\wedge ist(i_n,\phi_n)\supset ist(i,\phi)$. However, this encoding does not produce a PLC which is equivalent to the MCS. Below is a formal proof of this fact.

Let $\mathbb{BR}$ be the set of bridge rules between a set I of contexts with language $L_i = L_j$ (for $i, j \in I$). Let $\mathbb{LA} \subset \mathbb{W}$ be the set of lifting axioms among the contexts I expressed in a new context ε not in I. The notation $\Gamma \vdash_{\boldsymbol{br}} i:\phi$ stands for: $i:\phi$ is derivable from Γ in the MCS with the set I of contexts, no axioms, and the set ***br*** of bridge rules.

Theorem 3.6. *There is no transformation $la(.)$ from bridge rules to finite sets (or equivalently conjunctions) of lifting axioms such that, for any finite subset $\boldsymbol{br} \subseteq \mathbb{BR}$ of bridge rules*:

$$\begin{aligned}&i_1:\phi_1,\dots,i_n:\phi_n \vdash_{\boldsymbol{br}} i:\phi\\&\textit{if and only if}\\&\vdash_\varepsilon \bigwedge_{br\in\boldsymbol{br}} la(br) \supset \big(ist(i_1,\phi_1)\wedge\cdots\wedge ist(i_n,\phi_n)\supset ist(i,\phi)\big)\end{aligned} \qquad (2)$$

Lifting axioms are not the only possible *ist*-formulas. There are *ist*-formulas, as for instance $\neg ist(i,\phi)$ or $ist(i,\phi)\supset ist(j,\psi)\vee ist(k,\theta)$, which are not lifting axioms in Horn form but could be used to represent the compatibility relation formulated by bridge rules. So the question arises of whether bridge rules can be encoded by generic *ist*-formulas in some external context ε.

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Theorem 3.7. *There is a transformation* $a(.)$ *from finite sets* $\boldsymbol{br} \in \mathbb{BR}$ *of bridge rules to ist-axioms, and a context* ε *such that*:

$$\begin{array}{l} i_1 : \phi_1, \ldots, i_n : \phi_n \vdash_{\boldsymbol{br}} i : \phi \\ \textit{if and only if} \\ \vdash_\varepsilon a(\boldsymbol{br}) \supset ist(i_1, \phi_1) \wedge \cdots \wedge ist(i_n, \phi_n) \supset ist(i, \phi) \end{array} \qquad (3)$$

In the above theorem we have shown that MCSs with a finite number of contexts and with finite languages, can be represented via lifting axioms. However notice from the formal proof given in Appendix A that in embedding LMS/MCS into PLC, bridge rules are not directly translated into implications, as one could expect. For instance the bridge rules

$$\frac{1 : p}{2 : q} br_{12} \qquad \frac{2 : q}{1 : r} br_{21} \qquad (4)$$

are not translated in the axioms of the form $ist(1, p) \supset ist(2, q)$ and $ist(2, q) \supset ist(1, p)$ as shown in the proof of Theorem 3.6. The proof of Theorem 3.7 given in Appendix A, shows that the transformation a of the bridge rules (4) is not computed by a direct (syntactic) translation of each single bridge rule. Indeed, $a(\boldsymbol{br})$ is determined by enumerating all the LMS-models of (4) and by axiomatizing them in a PLC-formula. This is not a problem of our translation, indeed any alternative translation which is equivalent $a(\boldsymbol{br})$ with more than two contexts cannot be reduced to a set of horn lifting axioms.

4. Discussion of the formal results

In the previous sections, we proved some important theorems about PLC, LMS/MCS, and their relationship. The results can be summarized as follows:

1. satisfiability in PLC with partial vocabularies is equivalent to satisfiability in PLC with a complete vocabulary (Theorem 3.1);
2. PLC can be embedded into a particular class of MCS, called MPLC (Theorem 3.5);
3. LMS/MCS cannot be embedded in PLC using only lifting axioms for encoding bridge rules (Theorem 3.6);
4. under some important restrictions (including the hypothesis that all contexts have finite and homogeneous propositional languages), LMS/MCS can be embedded in PLC, but only if we allow also axioms which are not lifting axioms (Theorem 3.7).

The aim of this section is to discuss the impact of these theorems on the conceptual appropriateness of the two systems as formal theories of context and contextual reasoning.

4.1. Context-dependent vocabularies

One of the intuitions which is more generally accepted in the community of people working on context is that it must be possible to associate one distinct vocabulary to each context. Indeed, a vocabulary always presupposes an implicit ontology [20], and

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thus allowing context-dependent (partial, local) vocabularies is a way to fulfill the intuitive requirement that each context is a partial (its language can express facts only about some portion of the world) and approximate (its language encodes some level of granularity) representation of the world.

PLC and LMS/MCS provide different technical solutions to the idea of context-dependent vocabularies:

- PLC starts from a global vocabulary and then model-theoretically defines a context vocabulary via partial truth assignments: the vocabulary of a context κ is the intersection of the domains of interpretation of truth assignments for κ;
- LMS/MCS adopts a more radical approach, and assigns a distinct (formal) language to each context: the vocabulary of a context κ is the set of well-formed formulae that can be built from a distinct vocabulary.

At a first glance, the two approaches seem to be equivalent. However, there is an important difference. In PLC, if ϕ is a well-formed formula in the context sequence $\bar{\kappa}\kappa$, then the formula $ist(\kappa, \phi)$ must be a well-formed formula in $\bar{\kappa}$. This means that the vocabulary of the context $\bar{\kappa}$ depends, at least partially, on the vocabulary of the context $\bar{\kappa}\kappa$. In many applications, this property seems undesirable. Consider, for example, the application of PLC to distributed databases (this application was proposed, for example, in [23]): it is not always the case that in a context (representing a database *db*1) one is aware of all the objects and relations that can be expressed in another context (representing another database *db*2), as the two databases may have only partially overlapping vocabularies (and thus ontologies). Intuitively, this means that we don't write a formula like $ist(db1, R)$ in *db*2 if R is not in the vocabulary of *db*2. Unfortunately, for the properties of *ist* in PLC, such a step cannot be prevented.[9]

Unlike PLC, in LMS/MCS a distinct (and autonomous) language is associated to each context. Thus, the fact that ϕ is a well-formed formula of $L_{\kappa\kappa'}$ does not necessarily entail that $ist(\kappa', \phi)$ is a well-formed formula of the language L_κ (nor vice versa, in case one is under the flatness hypothesis). This is so because $ist(\kappa', \phi)$ is a propositional formula of L_κ, and its interpretation is defined with respect to the local models of L_κ (not to the local models of the language $L_{\kappa\kappa'}$). The fact that $ist(\kappa', \phi)$ is not a formula of $L_{\kappa\kappa'}$ simply means that one cannot impose any constraint on the interpretation of ϕ in $\kappa\kappa'$ and $ist(\kappa', \phi)$ in L_κ.

In short, we can conclude that vocabularies in PLC are not completely context-dependent (and Theorem 3.1 is an illustration of this fact). This cannot be changed in PLC, as this property is part of the logic itself (due to the properties of *ist*-formulae). On the contrary, if this property is desirable in some application, it can be modeled in LMS/MCS as an additional constraint on the definition of a LMS/MCS-model.

[9] The situation is even worse under the so-called "flatness" hypothesis, namely the hypothesis that context sequence $\bar{\kappa}\kappa$ coincides with the context κ. In this case, if in κ we state that ϕ is true (or false) in a context κ', then ϕ is necessarily a well-formed formula of $\kappa\kappa'$. In other words, if in a context we state that ϕ is true (or false) in some other context, we force ϕ to be expressible in the language of that context.

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4.2. Specifying facts that hold in a context

Another crucial feature of a formal theory of context is the possibility of specifying that a fact is true (holds) in a given context. Both PLC and LMS/MCS allow the specification of facts that hold in a context in two ways: *directly*, by explicitly listing the facts that are true at a given index (e.g., $db2 : R$); and *compositionally*, namely asserting new facts in a context by exploiting the relationships with other facts that hold in different contexts.

Once again, PLC and LMS/MCS follow different approaches to formalize this property. PLC formalizes the compositional specification via *lifting axioms*, namely formulae of the form:

$$\kappa_{ext} : ist(\kappa_1, \phi) \supset ist(\kappa_2, \phi) \tag{5}$$

Notice that lifting axioms are always asserted in some external context.

In LMS/MCS, the compositional specification is formalized (i) model-theoretically via compatibility relations, and (ii) proof-theoretically via bridge rules. For example, the fact that a formula ϕ is true in a context κ_2 if it is true in a context κ_1 corresponds to the following compatibility relation:

$$\text{for any } c \in C,\ \text{if } c \models \kappa_1 : \phi, \quad \text{then } c \models \kappa_2 : \phi \tag{6}$$

The corresponding bridge rule is:

$$\frac{\kappa_1 : \phi}{\kappa_2 : \phi} br_{(6)} \tag{7}$$

With respect to the general desiderata of a theory of context, the solution in PLC has two drawbacks. The first, which will be discussed in Section 4.5, is that lifting axioms alone are not expressive enough to encode all the relations that can be expressed via bridge rules (as we proved in Theorem 3.7). The second is a representational issue. As we said, lifting axioms can only be stated in an external context, which must be expressive enough to represent facts in both contexts (using *ist*-formulae); whereas, with bridge rules, one does not need to define an external context. Of course, there are situations in which having the external context may be an advantage (for example, it allows reasoning about lifting axioms themselves, and thus one can discover that certain lifting axioms are redundant, or lead to inconsistent contexts). However, in general, specifying an external context can be very costly—especially when there are many interconnected contexts—as the external context essentially duplicates all the information of each context. LMS/MCS allows both solutions. Indeed, instead of using bridge rules to lift a fact ϕ from κ_1 to κ_2, one can define a third context connected with κ_1 and κ_2 via bridge rules (see Definition 3.2) and explicitly add an axiom like (5) to this new context.[10]

4.3. Context-dependent truth

Another important requirement of a logic of context is that truth is context-dependent, namely the truth value of a fact must depend on the context in which it is asserted.

[10] This approach was used, for example, in the solution to the qualification problem presented in [6].

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Technically, this corresponds to requiring that truth (validity) be defined exclusively in terms of the truth assignments (local models) associated with each context. However, in PLC this is not precisely the case, as the satisfiability of *ist*-formulae does not depend on the assignments of the contexts in which they occur.

A PLC-model associates with each context sequence a set of evaluations for primitive propositions, which defines the truth of facts for that context. However, we can easily observe that the truth of a formula $ist(\kappa, \phi)$ depends on the truth assignments of the context κ. In fact, the following property holds:

$$\text{for any pair of assignment } \nu, \nu' \in \mathfrak{M}(\bar{\kappa}),\ \mathfrak{M}, \nu \models_{\bar{\kappa}} ist(\kappa, \phi) \\ \text{if and only if } \mathfrak{M}, \nu' \models_{\bar{\kappa}} ist(\kappa, \phi) \tag{8}$$

The property above shows that the truth value of $ist(\kappa, \phi)$ in $\bar{\kappa}$ is defined by the assignments of the context $\bar{\kappa}\kappa$, and thus the assignments of the context $\bar{\kappa}$ do not have any effect on its truth value. In short, the truth value of $ist(\kappa, \phi)$ in $\bar{\kappa}$ does not depend on the assignments of the context $\bar{\kappa}$, which makes its truth dependent on conditions which are independent from $\bar{\kappa}$.

4.4. *Describing relations between contexts*

The main feature of a formal theory of context, however, is the ability to formalize the relations existing between different contexts. To achieve this goal, PLC and LMS/MCS adopt two different strategies:

- PLC is based on a combination of lifting axioms plus the other axioms and rules for exiting and entering contexts;
- LMS/MCS is based on the mechanism of bridge rules.

Our goal is to argue that the second approach is more general, and intuitively more adequate, to model the desiderata of a theory of context. In particular, we stress that:

1. while lifting axioms can easily (and straightforwardly) be mapped onto bridge rules (or lifting axioms plus reflections rules, as shown in Definition 3.2), the converse is not true, which means that some relations formalized through bridge rule may require a complex translation into PLC which requires more than just a collection (any collection) of lifting axioms;
2. the properties of *ist* force PLC to embed in the logic a relation between contexts, expressed by the axiom (Δ), which seems hardly justifiable as a general relation between contexts.

The first point is a direct consequence of Theorem 3.6, in which we proved that the bridge rules (4) cannot be translated in axioms of the form $ist(1, p) \supset ist(2, q)$ and $ist(2, q) \supset ist(1, p)$. Theorem 3.7 shows that the translation is possible, but the needed axioms (A.12) are determined by enumerating all the LMS-models of (4) and by axiomatizing them in a formula of PLC. This does not depend on our translation, as

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any alternative translation which is equivalent to the axiom (A.12) with more than two contexts cannot be reduced to a set of lifting axioms. From the considerations above, we can conclude that in general LMS/MCS allows a simpler description of the relations between contexts.

The second remark is about the axiom (Δ), i.e., $ist(\kappa_1, ist(\kappa_2, \phi) \vee \psi) \supset ist(\kappa_1, ist(\kappa_2, \phi)) \vee ist(\kappa_1, \psi)$. Its validity is related to property (8). However, (Δ) does not seem to model a truly general relation between contexts, and therefore its status of logical axiom is quite dubious. Consider for example the following instance of (Δ):

$$ist\big(\kappa, ist(\kappa', \phi) \vee ist(\kappa', \psi)\big) \supset ist\big(\kappa, ist(\kappa', \phi)\big) \vee ist\big(\kappa, ist(\kappa', \psi)\big) \tag{9}$$

Intuitively, it says that it is never the case that a disjunctive fact about the truth in another context can hold without one of the disjuncts holding in that context. This principle, however, does not seem to hold in general. For example, it does not apply to belief contexts. Suppose κ represents the beliefs of an agent a, and $\kappa\kappa'$, the beliefs that a ascribes to another agent b. Suppose b flips a coin, but does not know whether it is head or tail. One would expect that a can ascribe to b the belief that it is head or tail, but not the belief that it is head nor the belief that it is tail. However, from the formula:

$$ist\big(a, ist(b, \text{Head}) \vee ist(b, \text{Tail})\big)$$

and the axiom (Δ) we can always infer:

$$ist\big(a, ist(b, \text{Head})\big) \vee ist\big(a, ist(b, \text{Tail})\big)$$

which intuitively is very implausible.

In LMS, the satisfiability of a formula of the type $ist(\kappa, \phi)$ is local to the context in which the formula is asserted (this is one of the distinguished properties of LMS in general), and therefore such a problem can be avoided. In order to prove the equivalence between MPLC and PLC, we had to impose a very strong compatibility relation such as condition 3 of Definition 3.3. However, it can be easily relaxed, as it is not part of the underlying logic.

4.5. Formalizing reasoning across contexts

In a formalization of context, it is very important to be able to represent logical consequence across different contexts, in order to adequately formalize reasoning across contexts. Indeed, logical consequence across different contexts formalizes the fact that a formula in a context is true as a consequence of the fact that other formulas are true in other contexts. This was one of the first requirements that McCarthy stated in his seminal paper on generality in AI [22], when he proposed contextual reasoning as an extension to natural deduction calculi, in which assumptions can be made in a context, the consequences can be derived in another context, and finally the conclusion can be derived in the original context.

An adequate calculus for a logic of context should not only formalize truth in contexts, but also allow assumption-based contextual truth. In other words a calculus should allow to infer that a formula is true in a context when other formulas (called assumptions) are true in other contexts.

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Let us see, for instance, how PLC and MCS/LMS represent the fact that ψ in κ is a logical consequence of ϕ in κ'. In PLC one needs a third ("top") context where logical consequence is represented by the formula $ist(\kappa, \phi) \supset ist(\kappa', \psi)$. Instead in MCS this is directly represented by the fact that $\kappa' : \psi$ is derivable via bridge rules from the assumption $\kappa : \phi$, i.e., that $\kappa : \phi \vdash_{\text{MCS}} \kappa' : \psi$.

If we generalize the previous example, we can see that in PLC logical consequence across a set I of contexts is represented via lifting axioms in an external context. This formalization, however, is not completely satisfactory for two main reasons:

- from a model-theoretic point of view, the truth condition for $ist(\kappa, \phi)$ (item 4 of Definition 2.2) formally interprets $ist(\kappa, \phi)$ as "ϕ is true in all the possible models (evaluations) of κ", i.e., ϕ is valid in κ, and not as ϕ is true in the current model (evaluation) of κ. Therefore, the lifting axiom

$$ist(\bar{\kappa}_1, \phi_1) \wedge \cdots \wedge ist(\bar{\kappa}_n, \phi_n) \supset ist(\bar{\kappa}, \phi) \tag{10}$$

  is interpreted as "if ϕ_h is valid in $\bar{\kappa}_h$, for each $1 \leqslant h \leqslant n$, then ϕ is valid in $\bar{\kappa}$" which formally differs from "if ϕ_h is *true* in $\bar{\kappa}_h$, for each $1 \leqslant h \leqslant n$, then ϕ is *true* in $\bar{\kappa}$";
- from a proof-theoretic point of view, in PLC, to infer the formula (10) one does not assert each ϕ_h in $\bar{\kappa}_h$ and infer the consequence ϕ in $\bar{\kappa}$, by navigating across contexts. Indeed, the reasoning pattern followed for proving (10) in PLC is: first lift up properties from each context $\bar{\kappa}_h$ and $\bar{\kappa}$ to ε, and then reason propositionally in ε.

In LMS/MCS, logical consequence is explicitly defined (item 7 of Definition 2.5). The fact that $\bar{\kappa} : \psi$ is a logical consequence of $\bar{\kappa}' : \phi$, w.r.t. a class of LMS-models $\mathfrak{C}$, is explicitly formalized as:

$$\bar{\kappa} : \phi \models_{\mathfrak{C}} \bar{\kappa}' : \psi \tag{11}$$

The MCS associated with the class of LMS-models $\mathfrak{C}$, provides an axiomatization of $\models_{\mathfrak{C}}$, based on Natural Deduction, which allows us to derive the formula $\bar{\kappa}' : \psi$ starting from the *assumption* $\bar{\kappa} : \phi$, whenever $\bar{\kappa} : \phi \models_{\mathfrak{C}} \bar{\kappa}' : \psi$.

5. Conclusions

This paper is the first attempt to provide a technical and conceptual comparison between PLC and LMS/MCS. Even though these two formalisms are perhaps the most significant attempts to provide a logic of context in AI, so far the comparison between them was limited to cross-references and a few lines of related works. We believe that the results presented in this paper will help clarify the technical and conceptual differences between the two approaches.

We stressed the fact that the two formalisms do not provide equivalent solutions, even if they share some of the intuitive motivations for having a formal theory of context in AI. The main results of the technical comparison are that (i) that PLC can be embedded into a particular class of MCS, called MPLC; (ii) that MCS cannot be embedded in PLC using only lifting axioms to encode bridge rules, and (iii) that, under some important restrictions

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(including the hypothesis that each context has finite and homogeneous propositional languages), MCS can be embedded in PLC, but only if we allow also axioms which are not lifting axioms. However, we argued that the restrictions needed to prove the second theorem have a significant impact on the appropriateness of PLC to capture the intuitive desiderata of a logic of context in AI.

Appendix A. Proof of theorems

Proof of Theorem 3.1. We prove the theorem by showing that each PLC-model $\mathfrak{M}$ can extended to a PLC-model $\mathfrak{M}_c$ with a complete vocabulary with the following property:

$$\text{For any formula } \phi \text{ and context sequence } \bar{\kappa}, \text{ such that } \mathsf{Vocab}(\phi, \bar{\kappa}) \in \mathsf{Vocab}(\mathfrak{M}),$$
$$\mathfrak{M} \models_{\bar{\kappa}} \phi \text{ iff } \mathfrak{M}_c \models_{\bar{\kappa}} \phi \tag{A.1}$$

The *completion* of a PLC-model $\mathfrak{M}$ is the PLC-model $\mathfrak{M}_c$ defined as follows. For any $\bar{\kappa} \in \mathbb{K}^*$:

- if $\mathfrak{M}(\bar{\kappa})$ is undefined, then $\mathfrak{M}_c(\bar{\kappa})$ contains all the possible total assignments to $\mathbb{P}$.
- if $\mathfrak{M}(\bar{\kappa})$ is defined, then $\mathfrak{M}_c(\bar{\kappa})$ is the following set of assignments:

$$\left\{\nu_c : \mathbb{P} \to \{\text{true}, \text{false}\} \mid \nu_c \text{ is a completion of some assignment } \nu \in \mathfrak{M}(\bar{\kappa})\right\}$$

  where ν_c is a *completion* of ν if and only if ν_c agree with ν on the domain of ν.

Clearly $\mathfrak{M}_c$ is a PLC-model. To prove property (A.1) we show by induction on the complexity of ϕ, that for any assignment $\nu \in \mathfrak{M}(\bar{\kappa})$, and for any completion ν_c of ν in $\mathfrak{M}_c$:

$$\mathfrak{M}, \nu \models_{\bar{\kappa}} \phi \quad \text{iff} \quad \mathfrak{M}_c, \nu_c \models_{\bar{\kappa}} \phi$$

Base case. $\mathfrak{M}, \nu \models_{\bar{\kappa}} p$ iff $\nu(p) = \text{true}$, and since any extension of ν_c agrees with ν on its domain, $\nu_c(p) = \text{true}$.

Step case. $\mathfrak{M}, \nu \models_{\bar{\kappa}} \neg\phi$ iff not $\mathfrak{M}, \nu \models_{\bar{\kappa}} \phi$, iff, by induction, not $\mathfrak{M}_c, \nu_c \models_{\bar{\kappa}} \phi$, iff $\mathfrak{M}_c, \nu_c \models_{\bar{\kappa}} \neg\phi$. The case of $\phi \supset \psi$ is similar. Let us consider the case of $ist(\kappa, \phi)$. $\mathfrak{M}, \nu \models_{\bar{\kappa}} ist(\kappa, \phi)$ iff for all $\nu' \in \mathfrak{M}(\bar{\kappa}\kappa)$, $\mathfrak{M}, \nu' \models_{\bar{\kappa}\kappa} \phi$, iff, by induction, for all $\nu'_c \in \mathfrak{M}_c(\bar{\kappa}\kappa)$, $\mathfrak{M}_c, \nu'_c \models_{\bar{\kappa}\kappa} \phi$, iff $\mathfrak{M}, \nu_c \models_{\bar{\kappa}} ist(\kappa, \phi)$. □

Proof of Theorem 3.3. Suppose that $c \models \bar{\kappa} : ist(\kappa, ist(\kappa', \phi) \vee \psi)$. If for all c', with $c_{\bar{\kappa}} = c'_{\bar{\kappa}}$, we have that $c' \models \bar{\kappa}\kappa : \psi$, then by condition 3 of Definition 3.2 of MBK($\mathbb{K}$)-model, we have that $c \models \bar{\kappa} : ist(\kappa, \psi)$ and therefore that $c \models \bar{\kappa} : ist(\kappa, ist(\kappa', \phi)) \vee ist(\kappa, \psi)$. If there is such a c', such that $c' \not\models \bar{\kappa}\kappa : \psi$, from the fact that, by condition 2 of Definition 3.2 of MBK($\mathbb{K}$)-model $c' \models \bar{\kappa}\kappa : ist(\kappa', \phi) \vee \psi$, we have that $c' \models \bar{\kappa}\kappa : ist(\kappa', \phi)$. By condition 4 of Definition 3.3 of MPLC-model, we have that $c' \models \bar{\kappa} : ist(\kappa, ist(\kappa', \phi))$. Since $c_{\bar{\kappa}} = c'_{\bar{\kappa}}$, then $c \models \bar{\kappa} : ist(\kappa, ist(\kappa', \phi))$, and therefore $c \models \bar{\kappa} : ist(\kappa, ist(\kappa', \phi)) \vee ist(\kappa, \psi)$.

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$$

Fig. A.1. A proof of Δ in MPLC.

Vice versa, suppose that $C \models \bar{\kappa}:(\Delta)$ and let us prove condition 4 of Definition 3.3. Since the formula $\bar{\kappa}: ist(\kappa, ist(\kappa',\phi) \vee \neg ist(\kappa',\phi)) \supset ist(\kappa, ist(\kappa',\phi)) \vee ist(\kappa, \neg ist(\kappa',\phi))$, is an instance of (Δ), and since $c \models \bar{\kappa}: ist(\kappa, ist(\kappa',\phi) \vee \neg ist(\kappa',\phi))$,

$$c \models \bar{\kappa}: ist\big(\kappa, ist(\kappa',\phi)\big) \vee ist\big(\kappa, \neg ist(\kappa',\phi)\big) \tag{A.2}$$

Suppose that $c \models \bar{\kappa}\kappa: ist(\kappa',\phi)$, then $c \not\models \bar{\kappa}\kappa: \neg ist(\kappa',\phi)$, and by condition 2 of Definition 3.2, $c \not\models \bar{\kappa}: ist(\kappa, \neg ist(\kappa',\phi))$. By property (A.2), and by the fact that $|c_{\bar{\kappa}}| \leqslant 1$, we have that $c \models \bar{\kappa}: ist(\kappa, ist(\kappa',\phi))$. □

Proof of Theorem 3.4. To prove soundness it is enough to prove that the unrestricted version of $\mathcal{R}_{up.}$, is sound w.r.t. logical consequence in MPLC-models. Namely that:

$$\bar{\kappa}\kappa: ist(\kappa',\phi) \models_{\mathrm{MPLC}} \bar{\kappa}: ist\big(\kappa, ist(\kappa',\phi)\big)$$

This is a trivial consequence of condition 4 of the definition of MPLC-model. Completeness of MPLC can be proved in an indirect way. We have indeed that MBK($\mathbb{K}$) is complete w.r.t. the class of MBK($\mathbb{K}$)-models. Furthermore, from Theorem 3.3, we have that, the class of MPLC-models, is the class of MBK($\mathbb{K}$)-models that satisfy $\bar{\kappa}:(\Delta)$. Completeness can be therefore proved by showing that (Δ) can be proved in MPLC. In Fig. A.1, we show a proof of (Δ). Notationally, $Prem(\Delta)$ and $Cons(\Delta)$ denote the premise and the consequence of (Δ) respectively. □

Proof of Theorem 3.5. Provability in PLC can be defined as provability in multi-modal K (denoted by $\vdash_K$) plus the axiom (Δ). For any subset $\mathbb{H}$ of $\mathbb{K}^*$, the notation $ist(\mathbb{H},\phi)$ denotes the set of formulae:

$$ist(\mathbb{H},\phi) = \big\{ ist\big(k_1, ist(k_2, \ldots ist(k_n,\phi))\big) \mid \kappa_1\kappa_2\ldots\kappa_n \in \mathbb{H} \big\}$$

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For any finite set of formulae $\Gamma = \{\gamma_1, \dots, \gamma_n\}$, $\bigwedge \Gamma$ denotes the formula $\gamma_1 \wedge \dots \wedge \gamma_n$. If $\vdash_{\bar\kappa} \phi$, then there is a *finite* set $\mathbb{H} \subseteq \mathbb{K}^*$, such that

$$\vdash_{\mathrm{K}} \bigwedge ist\big(\mathbb{H}, (\Delta)\big) \supset \phi$$

From the equivalence between multi-modal K and MBK($\mathbb{K}$) we have that

$$\vdash_{\mathrm{MBK}(\mathbb{K})} \bar\kappa : \bigwedge ist\big(\mathbb{H}, (\Delta)\big) \supset \phi$$

Since any formula in $\bar\kappa : ist(\mathbb{H}, (\Delta))$ is provable in MPLC, then we can conclude that

$$\vdash_{\mathrm{MPLC}} \bar\kappa : \phi$$

If $\nvdash_{\bar\kappa} \phi$, then we have that $\nvdash_{\varepsilon} \phi$, (where ε is the empty sequence). This implies that there is a model $\mathfrak{M}$, such that $\mathfrak{M} \not\models_{\varepsilon} \phi$. We define the MPLC-model $C_{\mathfrak{M}}$, that contains all the sequences c such that $c_{\bar\kappa} \in \mathfrak{M}(\bar\kappa)$, and $c_{\bar\kappa}$ is empty if $\mathfrak{M}$ is not defined for some $\bar\kappa'$, such that $\bar\kappa = \bar\kappa'\bar\kappa''$. It can be easily show that $C_{\mathfrak{M}}$ is a MPLC-model, and that $C_{\mathfrak{M}} \not\models \varepsilon : \phi$. □

Proof of Theorem 3.6. The theorem is proved by counterexample. Consider the two bridge rules in (4)

$$\frac{1:p}{2:q}br_{12} \qquad \frac{2:q}{1:r}br_{21} \tag{4}$$

where p, q, and r are three distinct propositional letters. Let br_{12} and br_{21} be both unrestricted (i.e., always applicable). Considering br_{12} or br_{21} separately, they do not affect theoremhood in either context 1 and 2. Formally, for $i = 1, 2$, $\vdash_{br_{12}} i : \phi$ if and only if ϕ is a propositional tautology, and analogously $\vdash_{br_{21}} i : \phi$ if and only if ϕ is a tautology (see [11,12] for a proof of a similar fact). However, combining br_{12} and br_{21} in the same MCS, new theorems, which are not tautologies, can be proved. An example of such a theorem is $1 : p \supset r$, and its proof is the following:

$$\cfrac{\cfrac{\cfrac{1:p^{(*)}}{2:q}\,br_{12}}{1:r}\,br_{21}}{1:p\supset r} \supset \mathrm{I}\ (\text{Discharging the assumption }^{(*)})$$

Let $la(br_{12})$ and $la(br_{21})$ be the following general conjunctions of lifting axioms:

$$la(br_{12}) = \bigwedge_{m=1}^{M} \left(\bigwedge_{k=1}^{K_m} ist(i_{mk}, \phi_{mk}) \supset ist(j_m, \psi_m) \right) \tag{A.3}$$

$$la(br_{21}) = \bigwedge_{n=M+1}^{N} \left(\bigwedge_{k=1}^{K_n} ist(i_{nk}, \phi_{nk}) \supset ist(j_n, \psi_n) \right) \tag{A.4}$$

where i_{mk}, i_{nk}, and j_n are either 1 or 2. Posing $\boldsymbol{br} = \{br_{12}, br_{21}\}$, we have that $\bigwedge_{br \in \boldsymbol{br}} la(br)$ is equivalent to the following formula:

$$\bigwedge_{n=1}^{N} \left(\bigwedge_{k=1}^{K_n} ist(i_{nk}, \phi_{nk}) \supset ist(j_n, \psi_n) \right)$$

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Suppose, by contradiction, that equivalence (2) holds. Since $1: p \supset r$ is derivable via br_{12} and br_{21}, we have that

$$\vdash_\varepsilon \bigwedge_{br \in \boldsymbol{br}} la(br) \supset ist(i, p \supset r) \tag{A.5}$$

Consider the PLC-model $\mathfrak{M}$ with $\mathfrak{M}(1)$ equal to all the assignments for L_1 and $\mathfrak{M}(2)$ equal to all the assignments for L_2. Since $p \supset r$ is not valid, there is an assignment ν such $\nu \not\models p \supset r$. By construction, $\mathfrak{M}(1)$ contains all the assignments to L_1. As a consequence $\mathfrak{M} \not\models_\varepsilon ist(1, p \supset r)$. Soundness of PLC and (A.5) entail that $\mathfrak{M} \not\models_\varepsilon \bigwedge_{br \in \boldsymbol{br}} la(br)$, and therefore, that there is an $n \leqslant N$ such that

$$\mathfrak{M} \models_\varepsilon \bigwedge_{k=1}^{K_n} ist(i_{nk}, \phi_{nk}) \quad \text{and} \quad \mathfrak{M} \not\models_\varepsilon ist(j_n, \psi_n) \tag{A.6}$$

The left part of (A.6) states that each ϕ_{nk} (with $1 \leqslant k \leqslant K_n$) is a tautology, as it must be true in all the assignments in $\mathfrak{M}(i_{nk})$. As a consequence we have that

$$\vdash_\varepsilon \bigwedge_{k=1}^{K_n} ist(i_{nk}, \phi_{nk}) \tag{A.7}$$

The right part of (A.6) states that there is an assignment $\nu \in \mathfrak{M}(j_n)$ such that $\nu \not\models \psi_n$, i.e., ψ_n is not a tautology. Let us consider two cases $n \leqslant M$, and $n > M$. In the first case, due to the definition of $la(br_{12})$, we have that

$$\vdash_\varepsilon la(br_{12}) \supset \left(\bigwedge_{k=1}^{K_n} ist(i_{nk}, \phi_{nk}) \supset ist(j_n, \psi_n) \right) \tag{A.8}$$

while, in the second one we have:

$$\vdash_\varepsilon la(br_{21}) \supset \left(\bigwedge_{k=1}^{K_n} ist(i_{nk}, \phi_{nk}) \supset ist(j_n, \psi_n) \right) \tag{A.9}$$

By applying Modus Ponens to (A.8) and (A.7), or to (A.9) and (A.7), we obtain one of the following two consequences:

$$\vdash_\varepsilon la(br_{12}) \supset ist(j_n, \psi_n) \quad \text{or} \quad \vdash_\varepsilon la(br_{21}) \supset ist(j_n, \psi_n)$$

If the equivalence holds we would have that, either $\vdash_{br_{12}} j_n : \psi_n$ or $\vdash_{br_{21}} j_n : \psi_n$, while ψ_n is not a tautology. But this is a contradiction. □

Proof of Theorem 3.7. The proof is constructive, i.e., we define the transformation $a(.)$ for each set of bridge rules. The definition of $a(\boldsymbol{br})$ passes through a syntactic encoding of the LMS-models for $\boldsymbol{br}$.

Let C be a LMS-model (i.e., a set of chains), the set of PLC-models $\mathfrak{M}_C$ corresponding to C is defined as follows:

$$\mathfrak{M}_C = \left\{ \mathfrak{M}_{C'} \mid C' \text{ is a subset of } C \text{ such that for any } i \in I,\ \mathfrak{M}(i) = \bigcup_{c \in C'} c_i \right\} \tag{A.10}$$

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Let $\mathbf{C}$ be the set of LMS-models for ***br***. The set $\mathfrak{M}_\mathbf{C}$ is defined as $\bigcup_{C\in\mathbf{C}}\mathfrak{M}_C$. Let us prove that the logical consequence defined by $\mathbf{C}$ can be represented by valid formulas in the set of models $\mathfrak{M}_\mathbf{C}$, i.e., that:

$$\begin{aligned}&i_1:\phi_1,\ldots,i_n:\phi_n \models_\mathbf{C} i:\phi\\ &\text{if and only if for all } \mathfrak{M}\in\mathfrak{M}_\mathbf{C}\\ &\mathfrak{M}\models_\varepsilon ist(i_1,\phi_1)\wedge\cdots\wedge ist(i_n,\phi_n)\supset ist(i,\phi)\end{aligned}\tag{A.11}$$

Suppose that $i_1:\phi_1,\ldots,i_n:\phi_n \models_\mathbf{C} i:\phi$. Let $\mathfrak{M}_{C'}\in\mathfrak{M}_\mathbf{C}$, with $C'\subseteq C\in\mathbf{C}$. Suppose that $\mathfrak{M}_{C'}\models_\varepsilon ist(i_k,\phi_k)$ for any $1\leqslant k\leqslant n$. This implies that for all $c\in C'$, $c_{i_k}\models\phi_k$. From the hypothesis we have that $c_i\models\phi$, and therefore that $\mathfrak{M}_{C'}\models_\varepsilon ist(i,\phi,)$.

Vice versa, let us prove that $\mathfrak{M}\models_\varepsilon ist(i_1,\phi_1)\wedge\cdots\wedge ist(i_n,\phi_n)\supset ist(i,\phi)$ for all $\mathfrak{M}\in\mathfrak{M}_\mathbf{C}$ implies that for any model C of ***br*** and for any chain $c\in C$, if $c_{i_k}\models\phi_k$ for $1\leqslant k\leqslant n$, then $c_i\models\phi$. Notice that, for any $c\in C\in\mathbf{C}$ we have that $\mathfrak{M}_{\{c\}}\in\mathfrak{M}_\mathbf{C}$. By definition (see Eq. (A.10)), $\mathfrak{M}_{\{c\}}$ is such that $\mathfrak{M}(i)=c_i$. By hypothesis we have that $\mathfrak{M}_{\{c\}}\models ist(i_1,\phi_1)\wedge\cdots\wedge ist(i_n,\phi_n)\supset ist(i,\phi)$, which implies that if $c_{i_k}\models\phi_k$ for all $1\leqslant k\leqslant n$, then $c_i\models\phi$.

To define $a(\boldsymbol{br})$ we proceed as follows: for any PLC-model $\mathfrak{M}\in\mathfrak{M}_\mathbf{C}$ we find a formula $\phi_\mathfrak{M}$, that axiomatizes exactly $\mathfrak{M}$. Then the axiomatization of $\mathfrak{M}_\mathbf{C}$ can be obtained by the disjunction of all the axiomatization $\phi_\mathfrak{M}$ associated to each single PLC-model $\mathfrak{M}$ of $\mathfrak{M}_\mathbf{C}$ (this definition is possible because $\mathfrak{M}_\mathbf{C}$ is finite).

Let $\mathfrak{M}\in\mathfrak{M}_\mathbf{C}$, and let $\phi_\mathfrak{M}$ be the following formula

$$\bigwedge_{i\in I}\left(ist\left(i,\bigvee_{\nu\in\mathfrak{M}(i)}\phi_\nu\right)\wedge\bigwedge_{\nu\in\mathfrak{M}(i)}\neg ist(i,\neg\phi_\nu)\right)\tag{A.12}$$

where ϕ_ν is the conjunction of all the literals verified by the assignment ν. (A.12) is a finite formula, for the set I of context is finite and the set of literals in each context is finite too. By adding (A.12) as axioms in the context ε we obtain an PLC that is satisfied only by the model $\mathfrak{M}$. Let

$$a(\boldsymbol{br})=\bigvee_{\mathfrak{M}\in\mathfrak{M}_\mathbf{C}}\phi_\mathfrak{M}$$

Let us now prove the equivalence (3). By soundness and completeness of ***br***, $i_1:\phi_1,\ldots,i_n:\phi_n\vdash_{\boldsymbol{br}} i:\phi$ holds if and only if

$$i_1:\phi_1,\ldots,i_n:\phi_n\models_\mathbf{C} i:\phi\tag{A.13}$$

By (A.11), we have that (A.13) holds if and only if for all $\mathfrak{M}\in\mathfrak{M}_\mathbf{C}$,

$$\mathfrak{M}\models_\varepsilon ist(i_1,\phi_1)\wedge\cdots\wedge ist(i_n,\phi_n)\supset ist(i,\phi)\tag{A.14}$$

By construction of $a(\boldsymbol{br})$, $\mathfrak{M}\models_\varepsilon a(\boldsymbol{br})$, if and only if $\mathfrak{M}\in\mathfrak{M}_\mathbf{C}$. This implies that (A.14) holds if and only if

$$\models_\varepsilon a(\boldsymbol{br})\supset ist(i_1,\phi_1)\wedge\cdots\wedge ist(i_n,\phi_n)\supset ist(i,\phi)\tag{A.15}$$

Finally, soundness and completeness of PLC implies that (A.15) holds if and only if $\vdash_\varepsilon a(\boldsymbol{br})\supset ist(i_1,\phi_1)\wedge\cdots\wedge ist(i_n,\phi_n)\supset ist(i,\phi)$, which concludes our proof. □

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